

# *Synchronizing Strongly Connected Digraphs*

M. V. Volkov

Ural State University, Ekaterinburg, Russia



## ***New Perspectives***

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This means that now we can address **many questions** previously studied for synchronizing automata to strongly connected primitive digraph with constant out-degree. It creates a unique opportunity for new developments in the area and also may shed some new light on longstanding open problems.

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- Complexity of calculating short reset word for **graphs**: given a strongly connected primitive digraph  $\Gamma$  with constant out-degree and a positive integer  $n$ , is there a reset word of length  $n$  for a synchronizing coloring of  $\Gamma$ ? Clearly is in NP but may even be in P.

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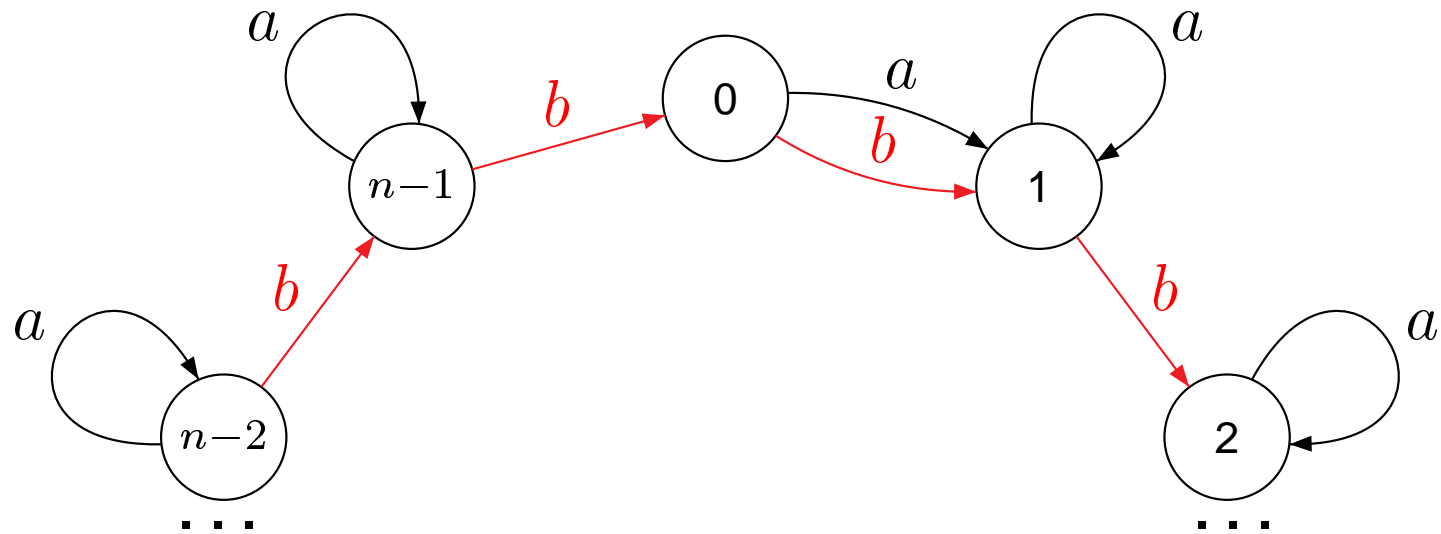
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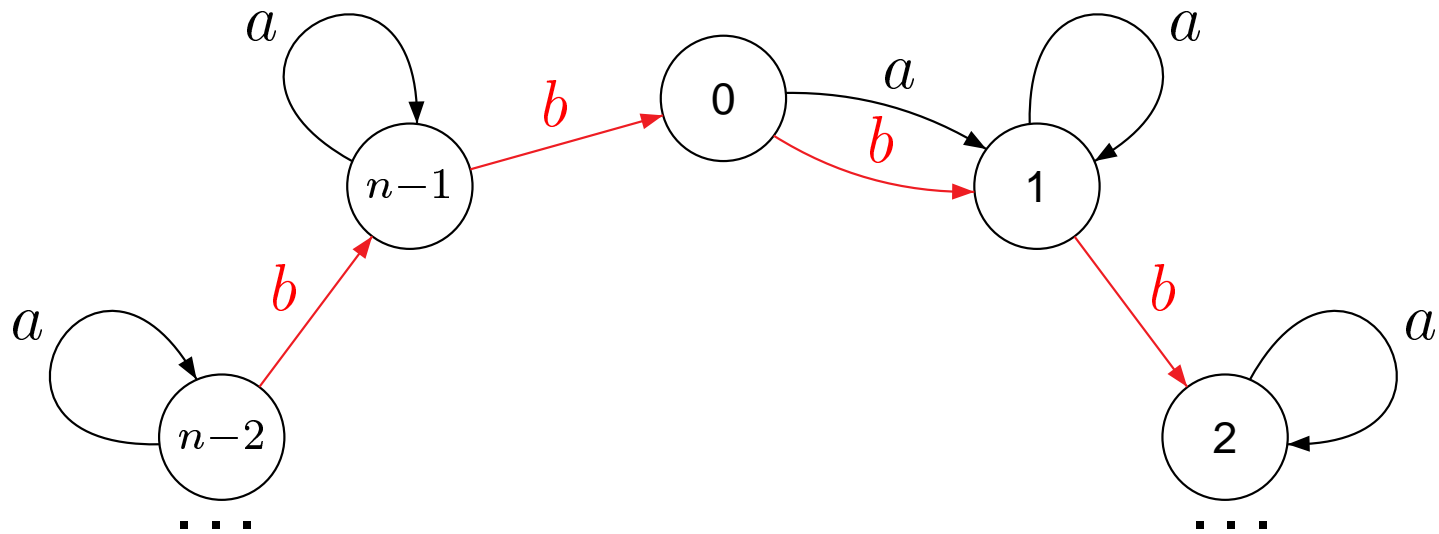
For instance, the Černý automata  $\mathcal{C}_n$ ,  $n = 2, 3, \dots$ , admit synchronizing recolorings with pretty short reset words, as already mentioned in Trahtman's lecture.

# Černý Automata



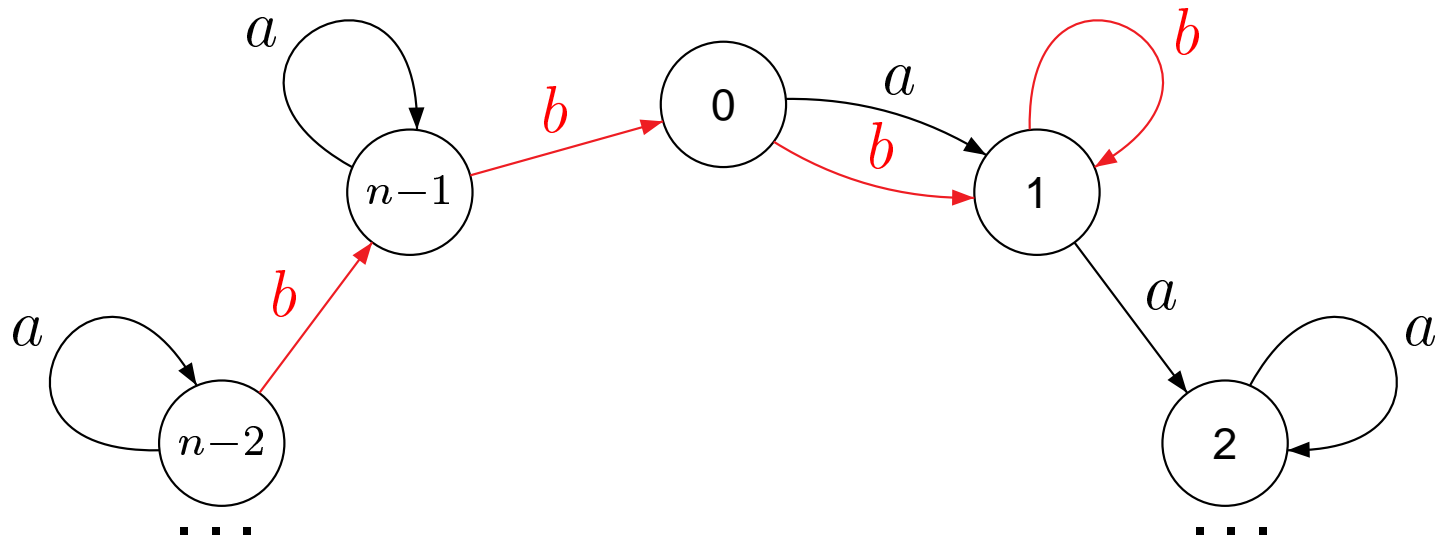
Recall that the states of  $\mathcal{C}_n$  are the residues modulo  $n$ , the input letters  $a$  and  $b$  act as follows:  $\delta(0, a) = 1$ ,  $\delta(m, a) = m$  for  $0 < m < n$ ,  $\delta(m, b) = m+1 \pmod{n}$ .

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However, one can recolor the same graph as shown in the picture, and the recolored automaton is reset by the word  $b^{n-1}$ .

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We present here all known proper synchronizing automata beyond the Černý series  $\mathcal{C}_n$ ,  $n = 3, 4, \dots$ , that reach the Černý bound. (A synchronizing automaton  $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$  is *proper* if none of the automata obtained from  $\mathcal{A}$  by erasing any letter in  $\Sigma$  are synchronizing.)

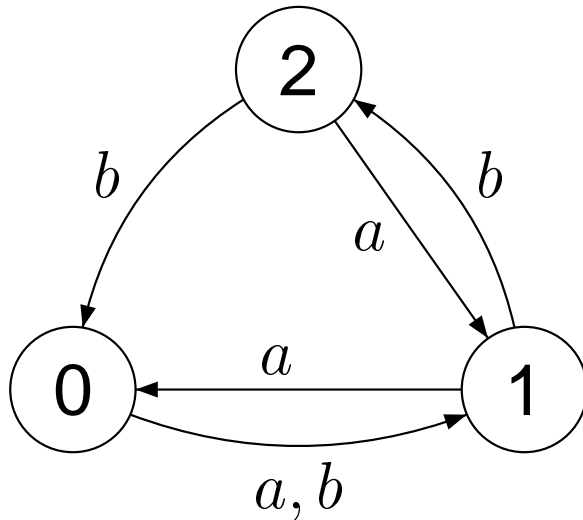
# *Sporadic Examples*

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For  $n = 3$  we have three sporadic automata:

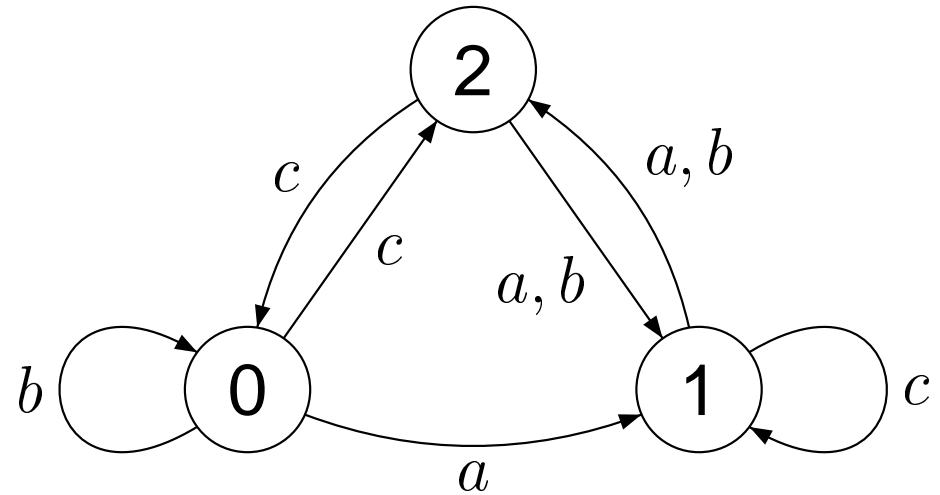
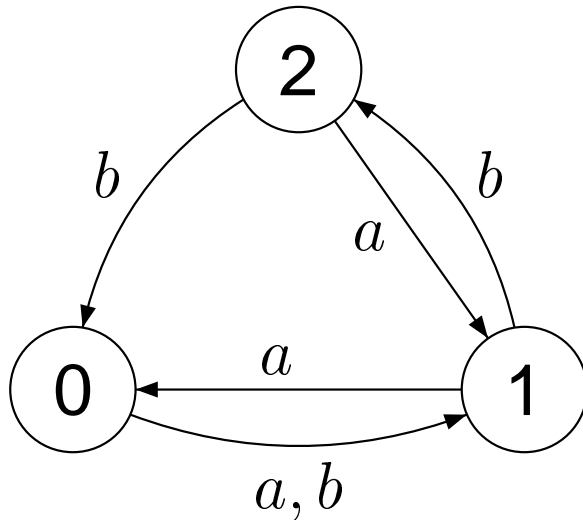
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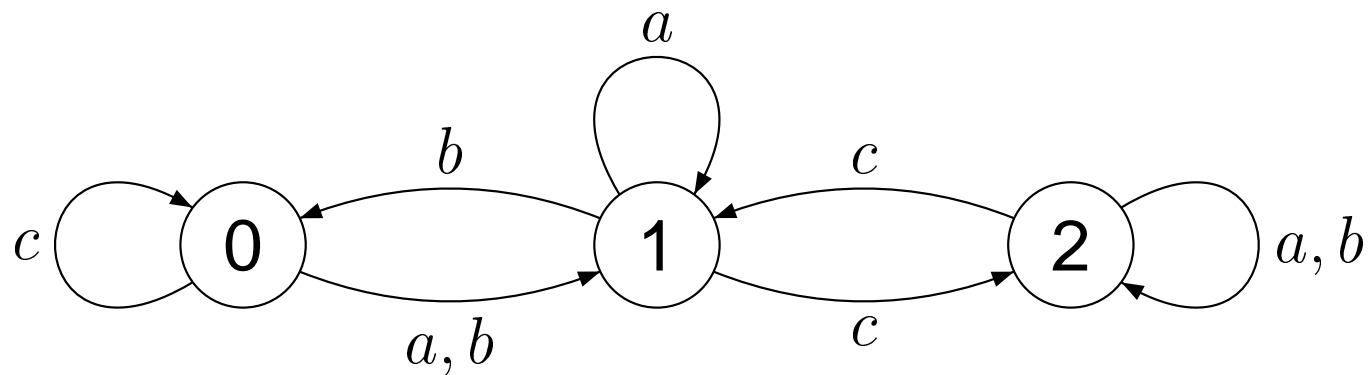
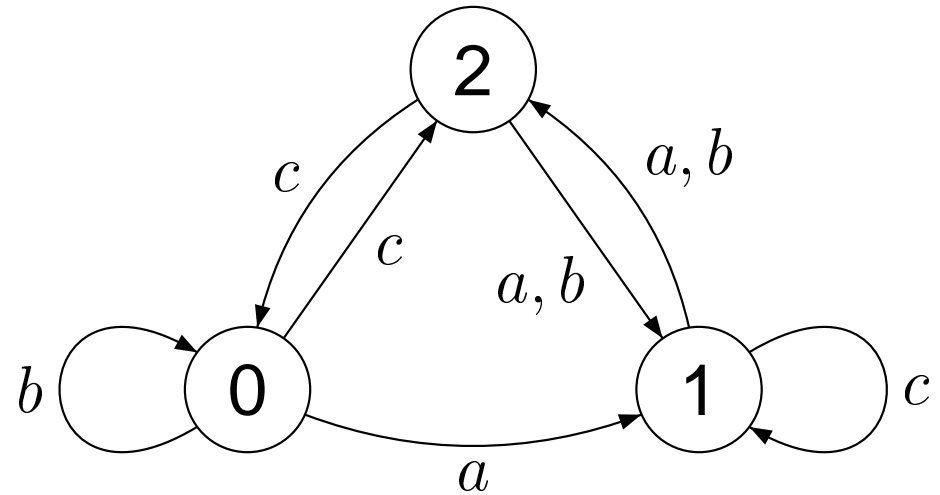
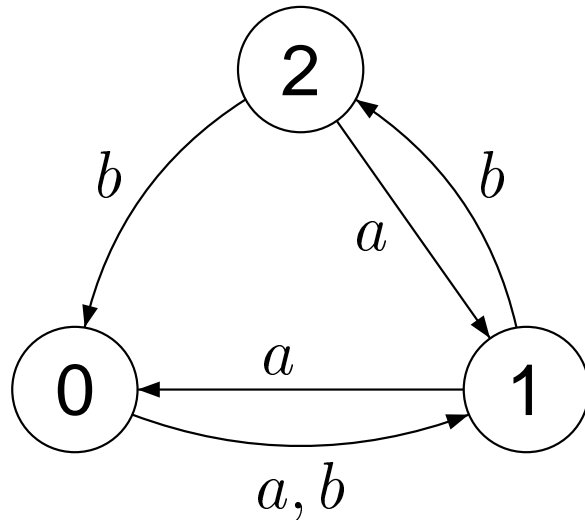
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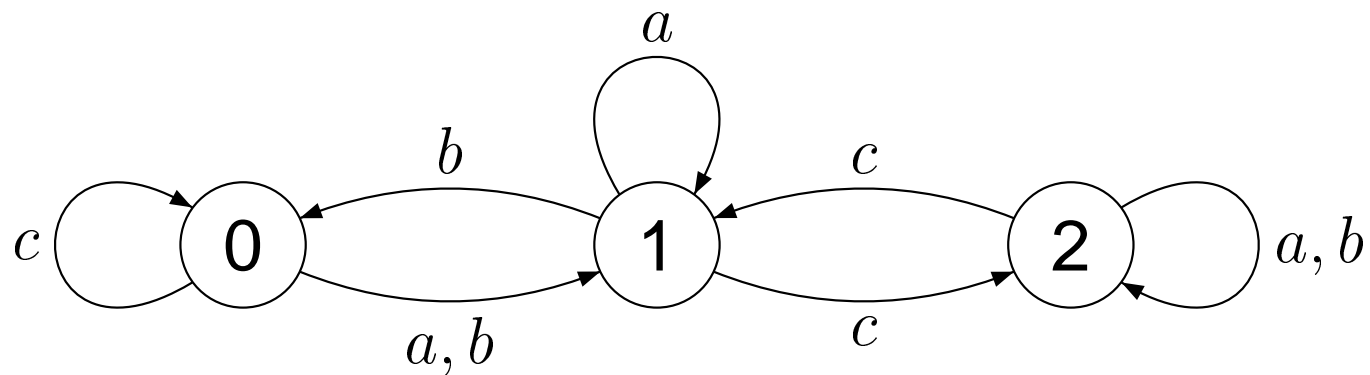
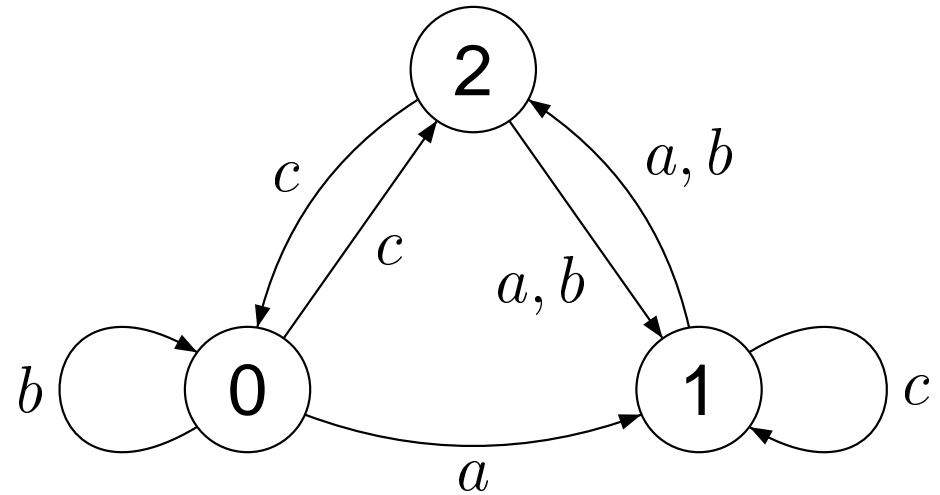
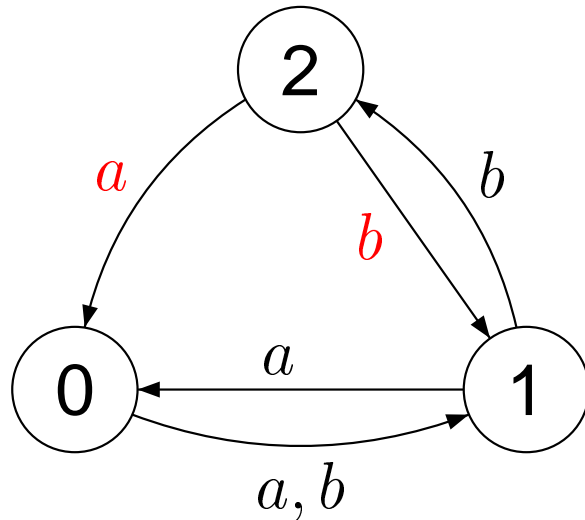
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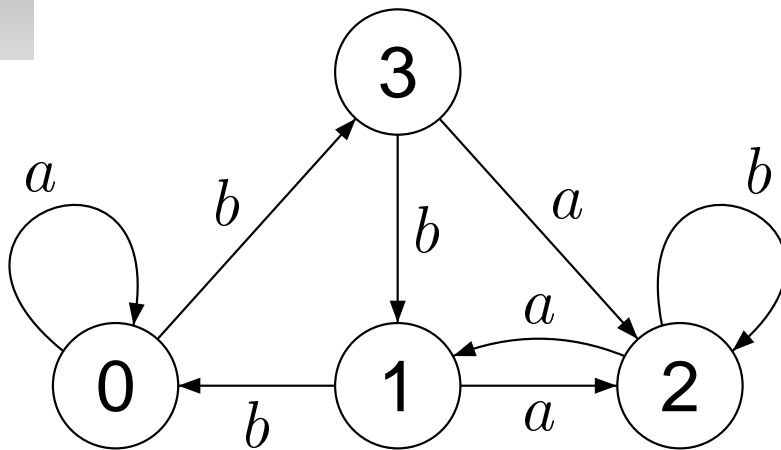
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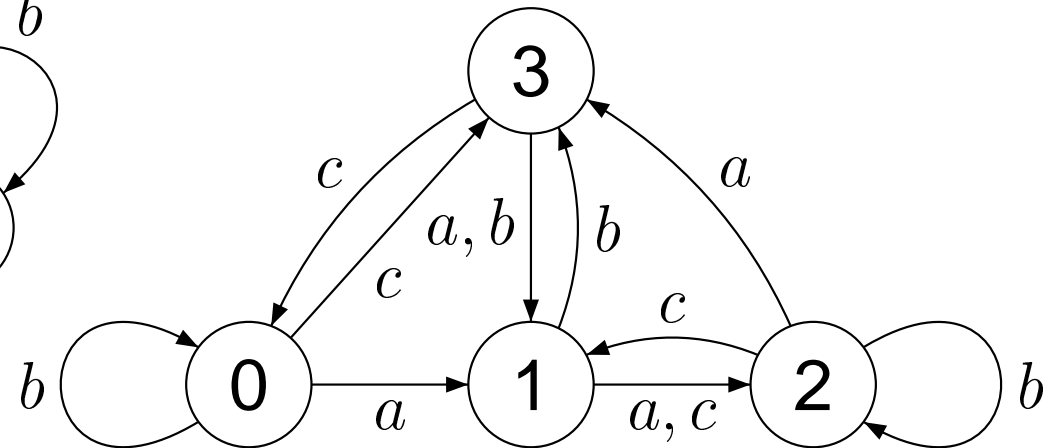
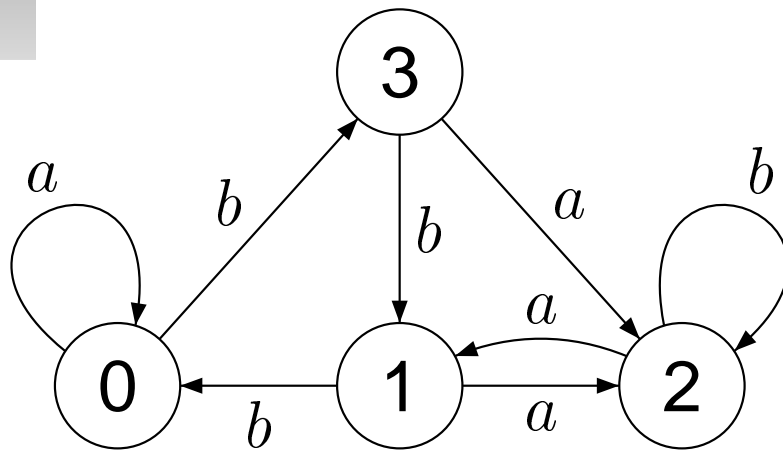
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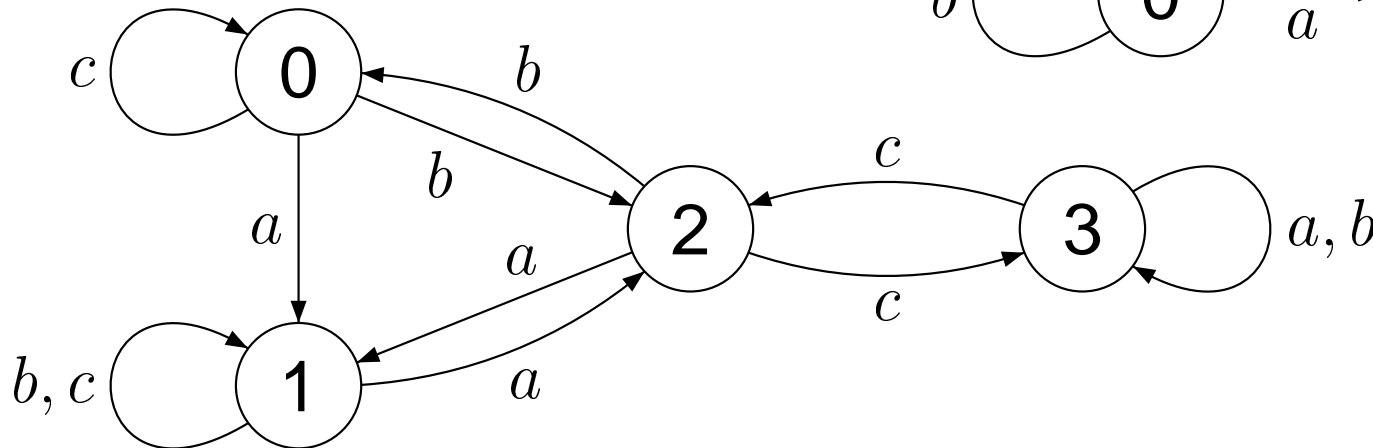
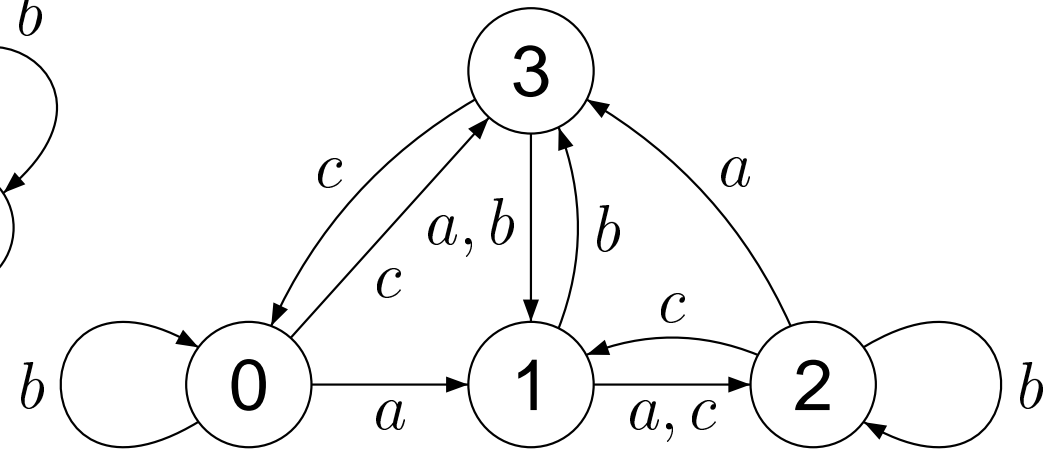
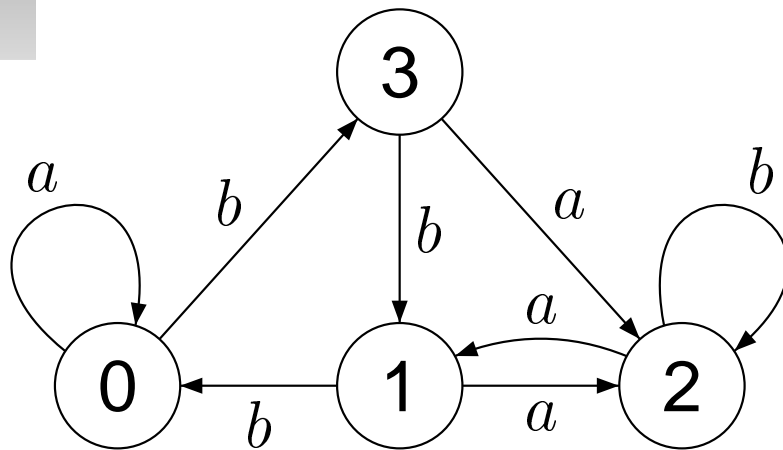
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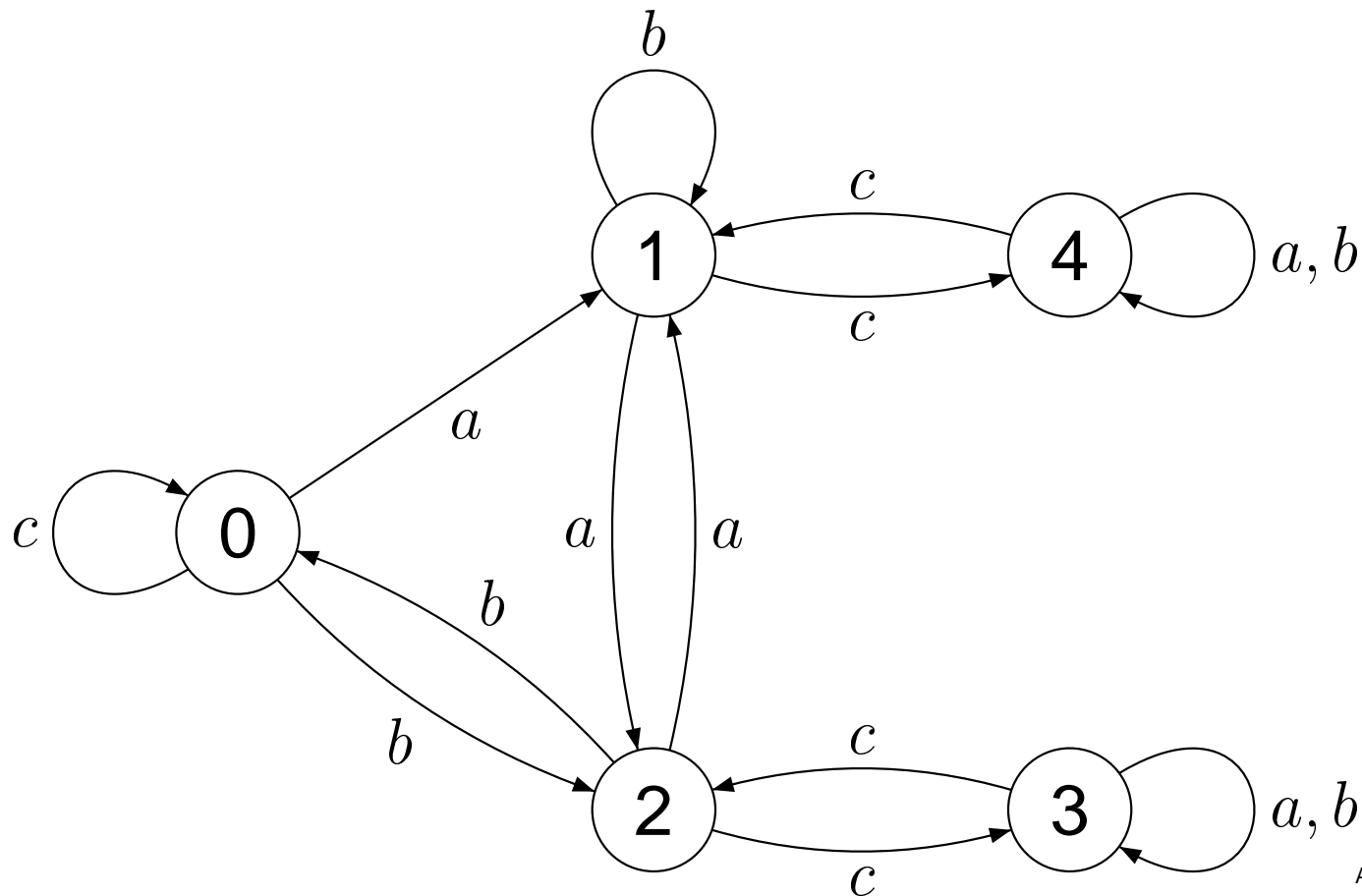
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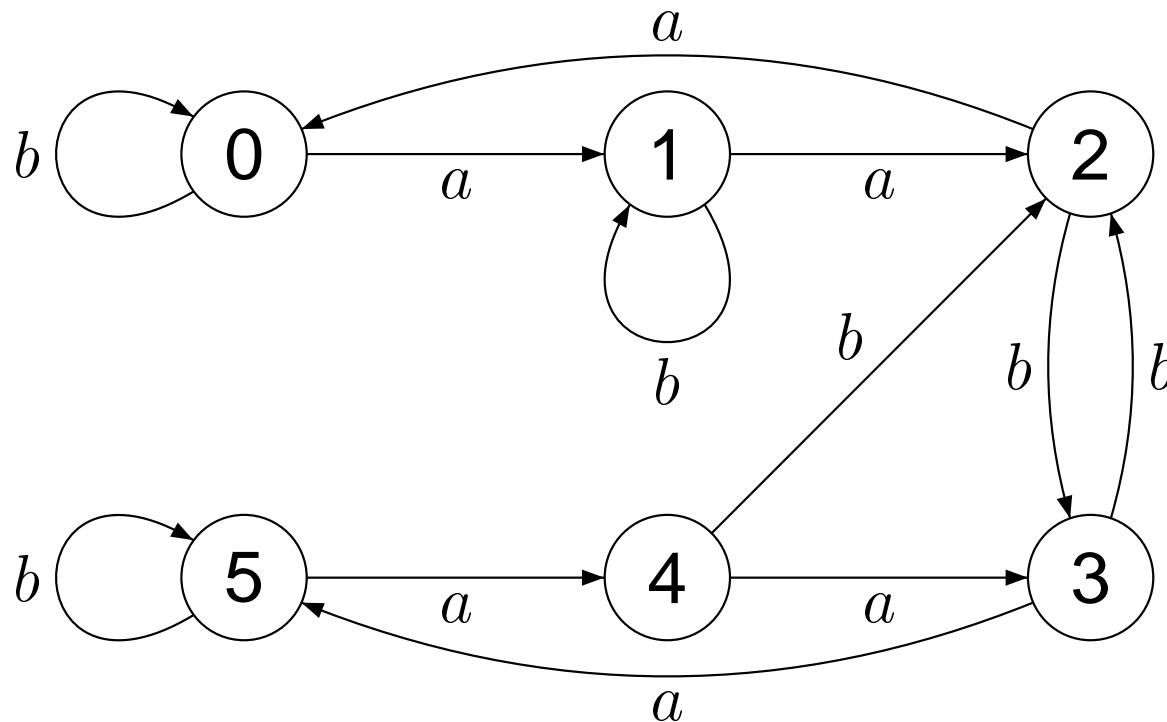
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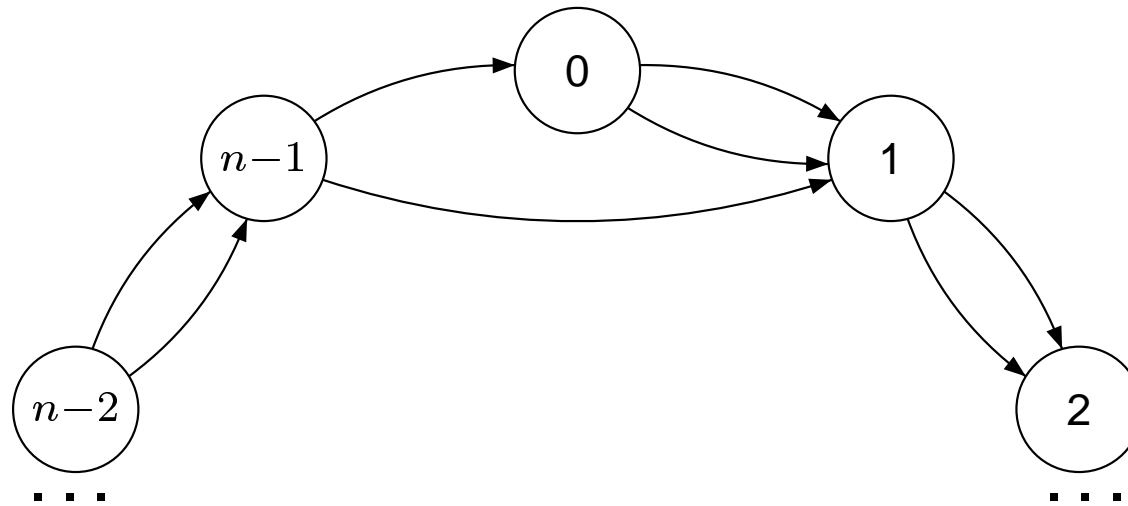
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**No**, a counterexample follows.

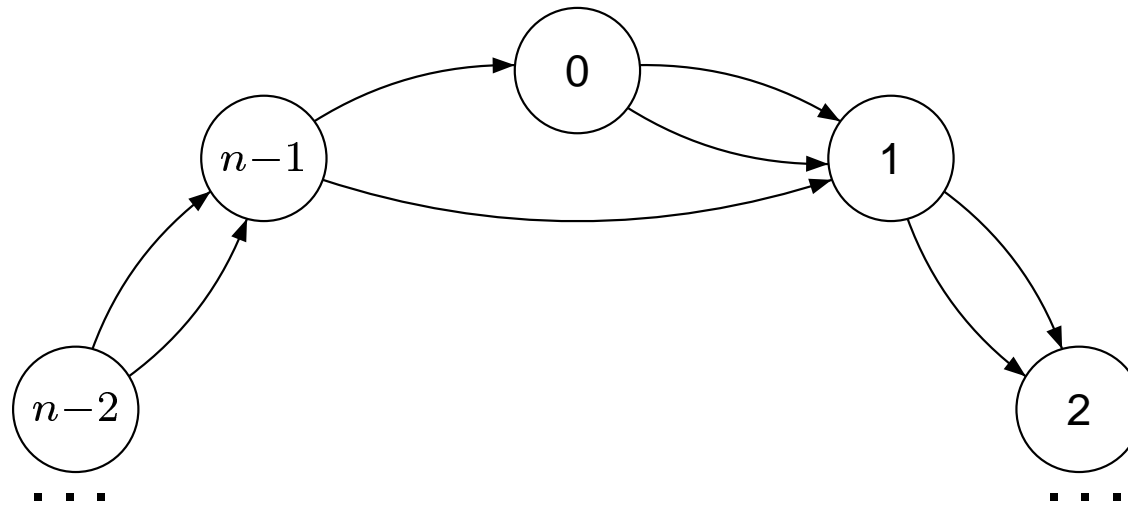
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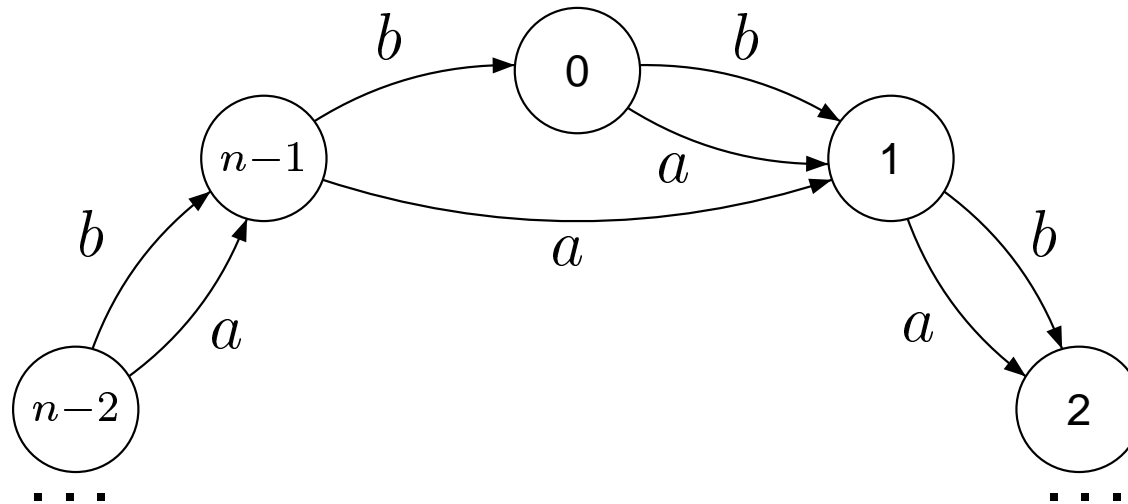
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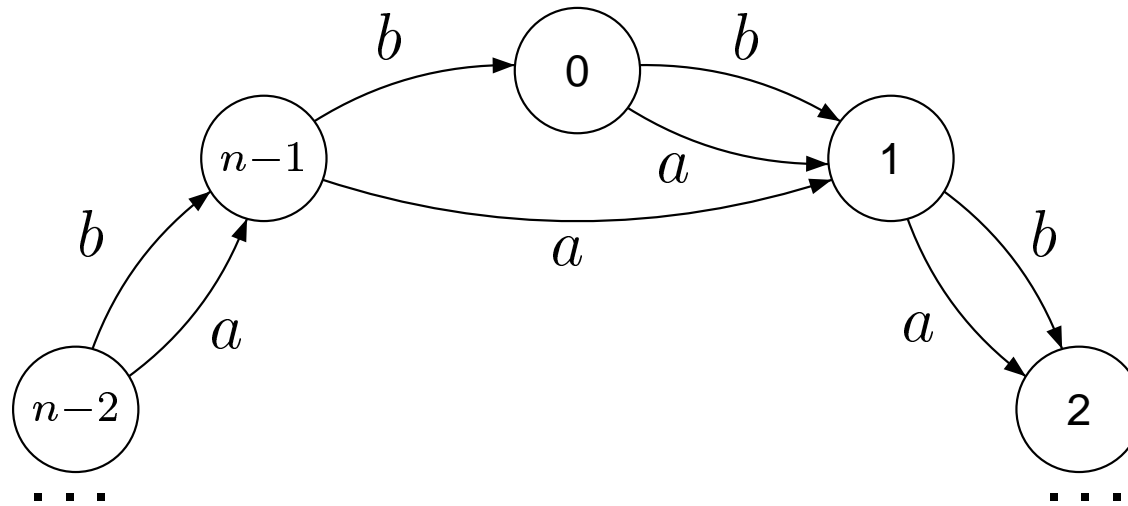
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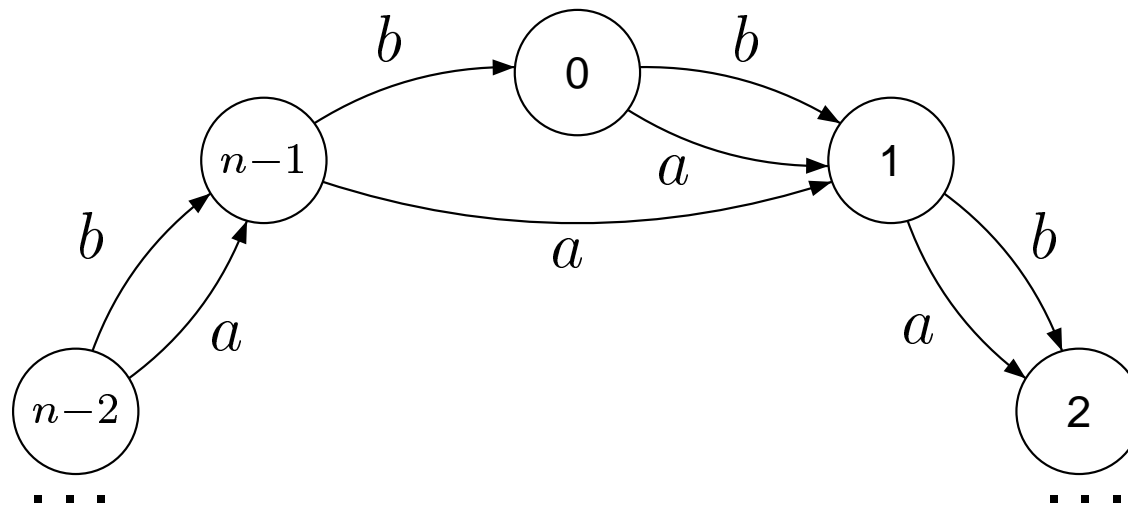
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The answer is  $n^2 - 3n + 3 = (n - 1)(n - 2) + 1$  and there is a unique reset word of this length, namely,  $(ab^{n-2})^{n-2}a$ .

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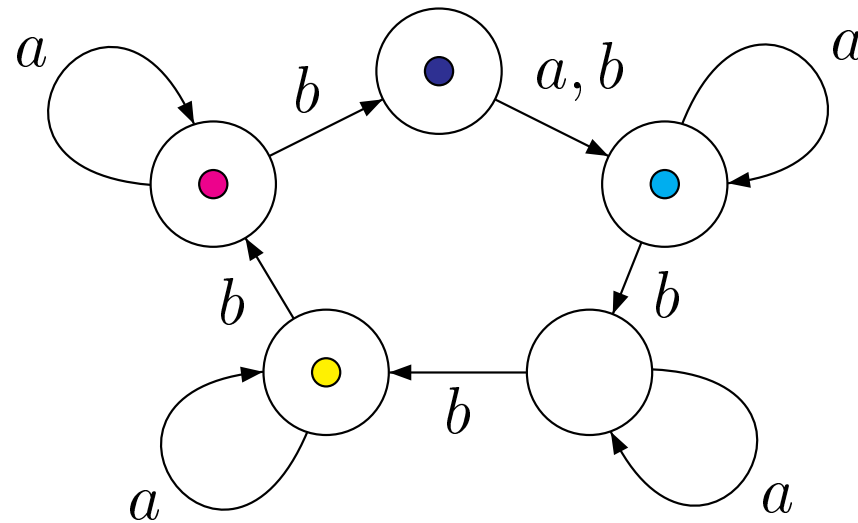
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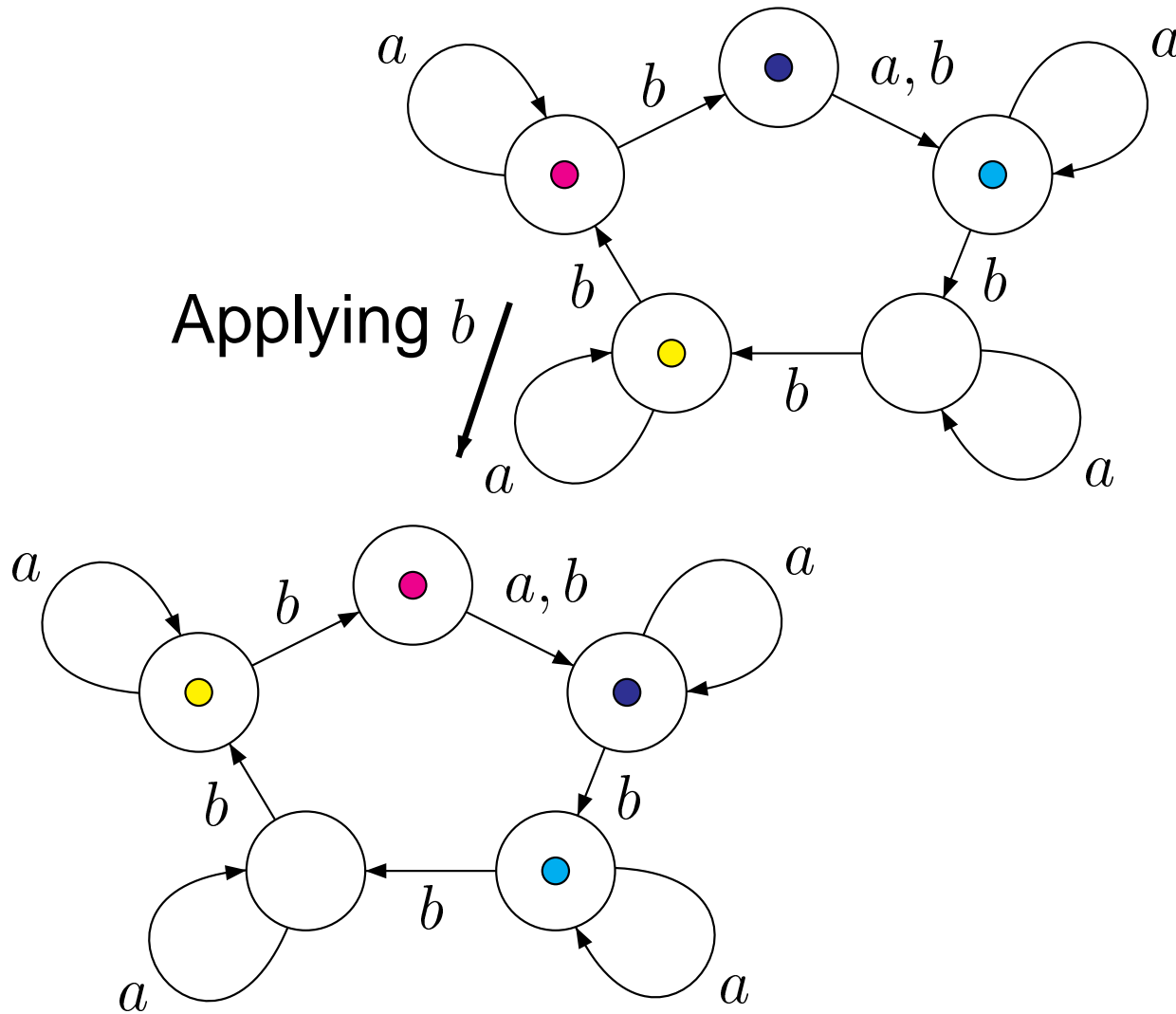
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- The only coin that remains at the end of the game is the **golden** coin  $G$ .

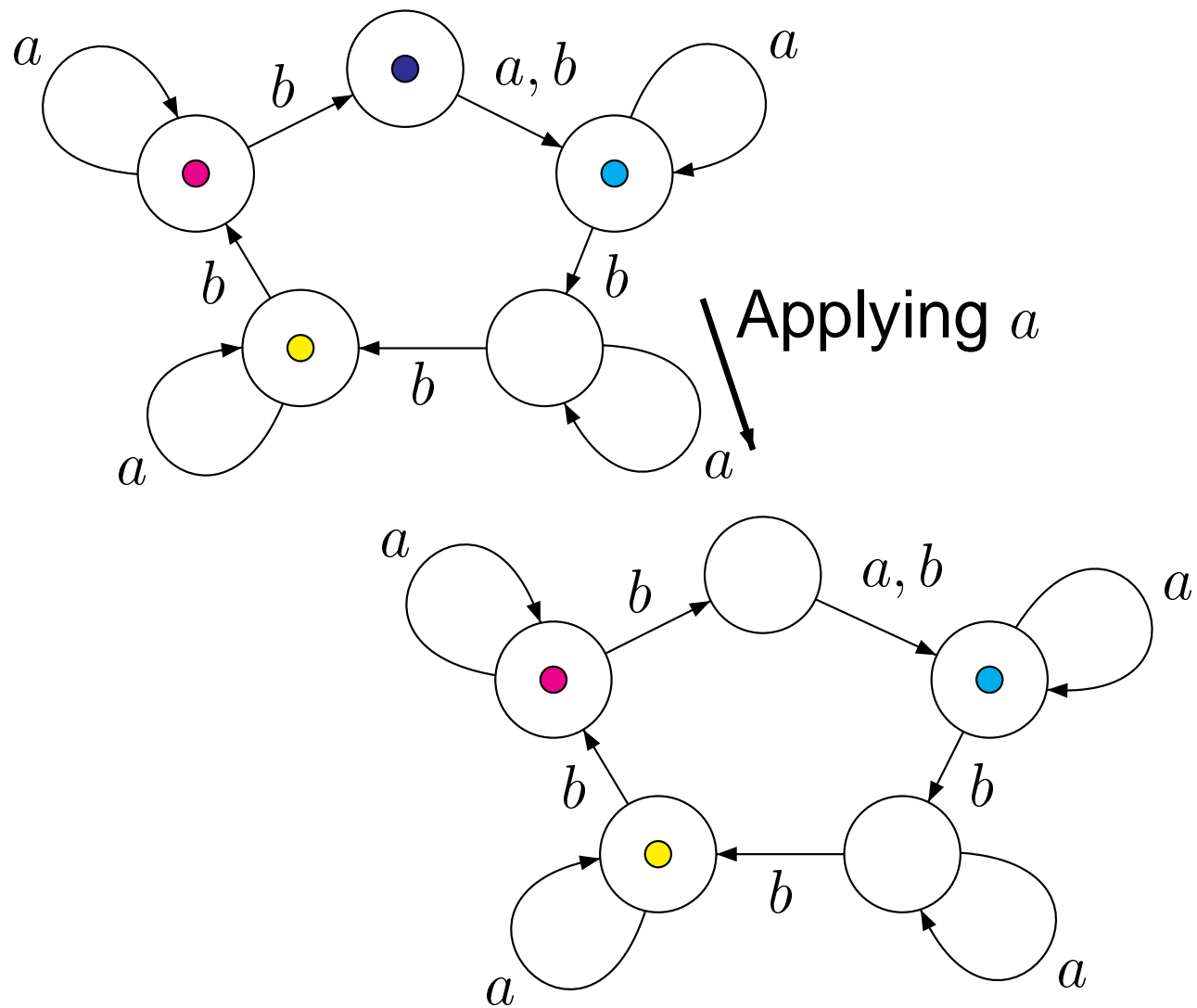
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- (i)  $\text{wg}(P_0) \geq n(n - 1)$  and  $\text{wg}(P_{|w|}) \leq n - 1$ ;
- (ii) for each  $i = 1, \dots, |w|$ , the action of the  $i^{\text{th}}$  letter of  $w$  decreases the weight by 1 at most, that is,  $1 \geq \text{wg}(P_{i-1}) - \text{wg}(P_i)$ .

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$$\text{Then } |w| = \sum_{i=1}^{|w|} 1 \geq \sum_{i=1}^{|w|} (\text{wg}(P_{i-1}) - \text{wg}(P_i)) =$$

$$\text{wg}(P_0) - \text{wg}(P_{|w|}) \geq n(n - 1) - (n - 1) = (n - 1)^2.$$

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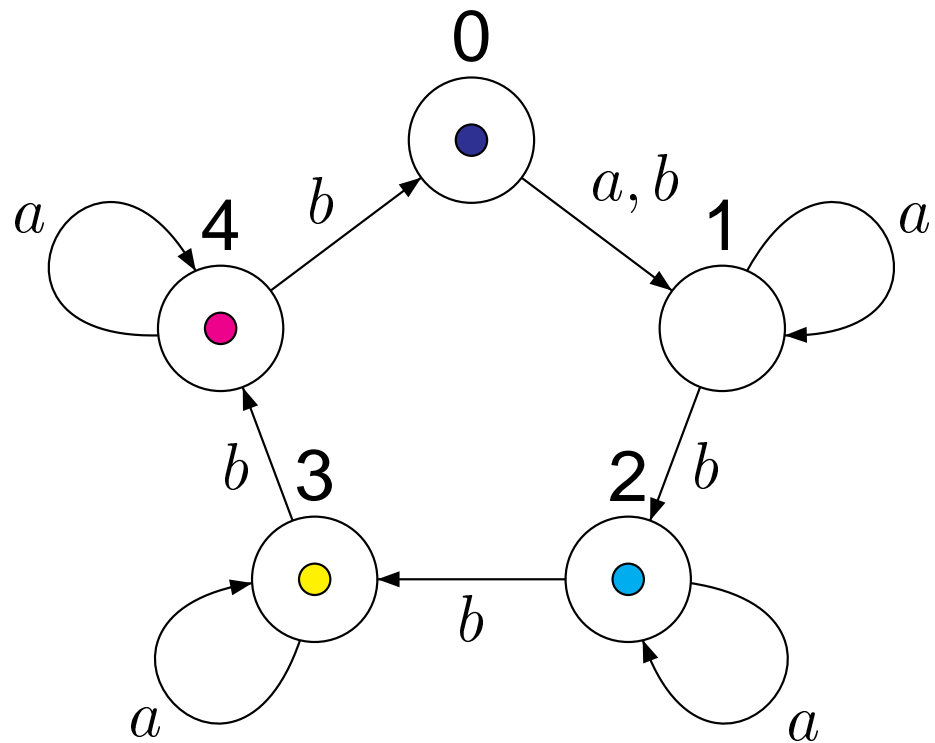
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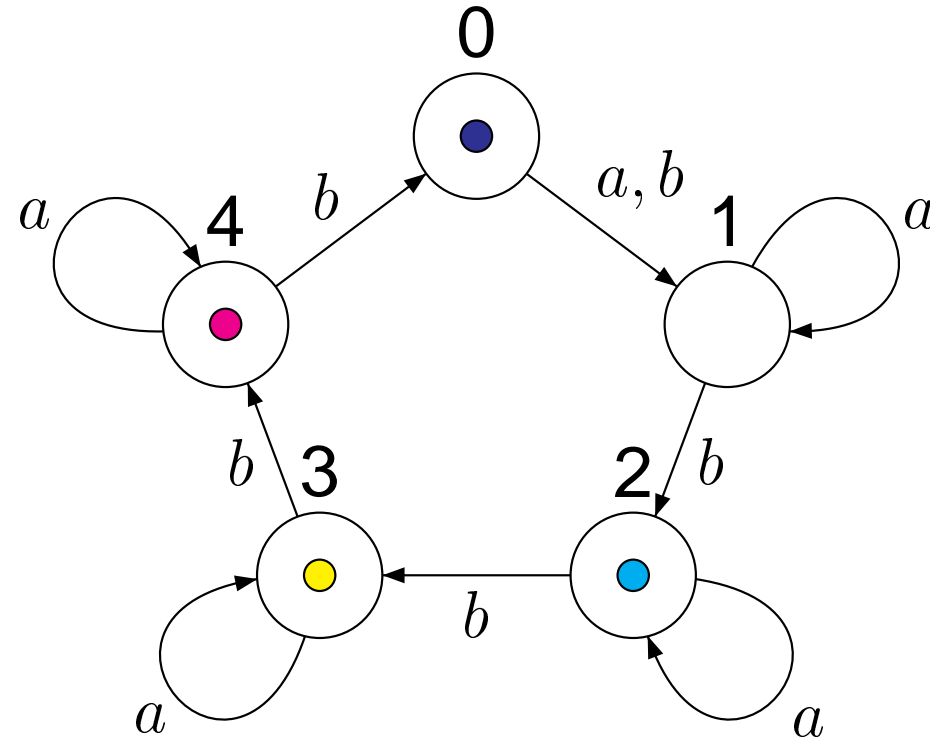
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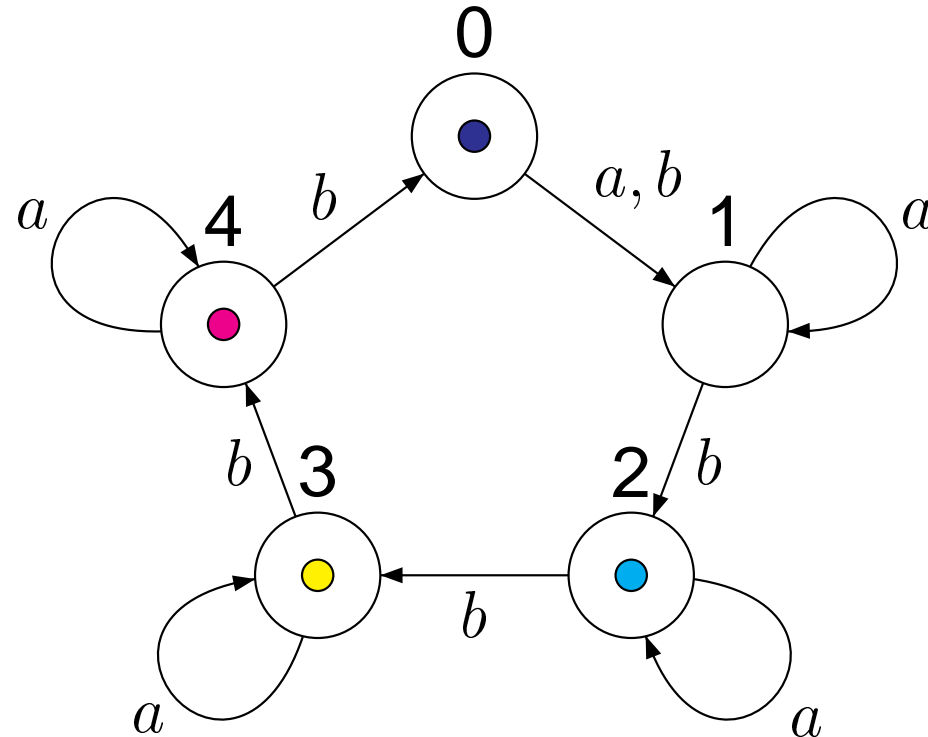


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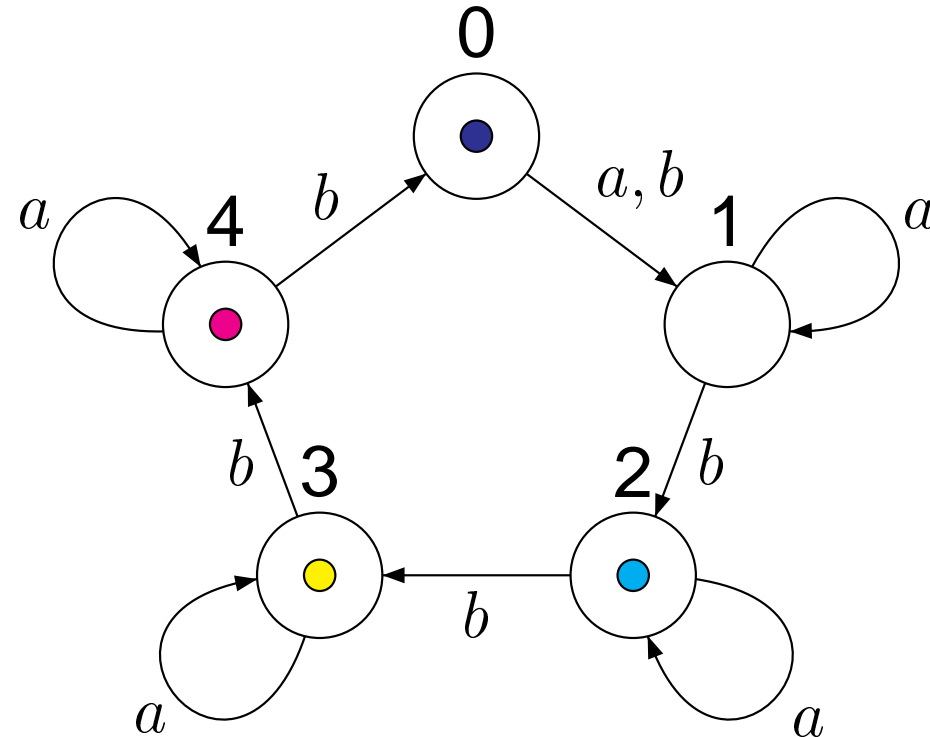
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# Černý Automata



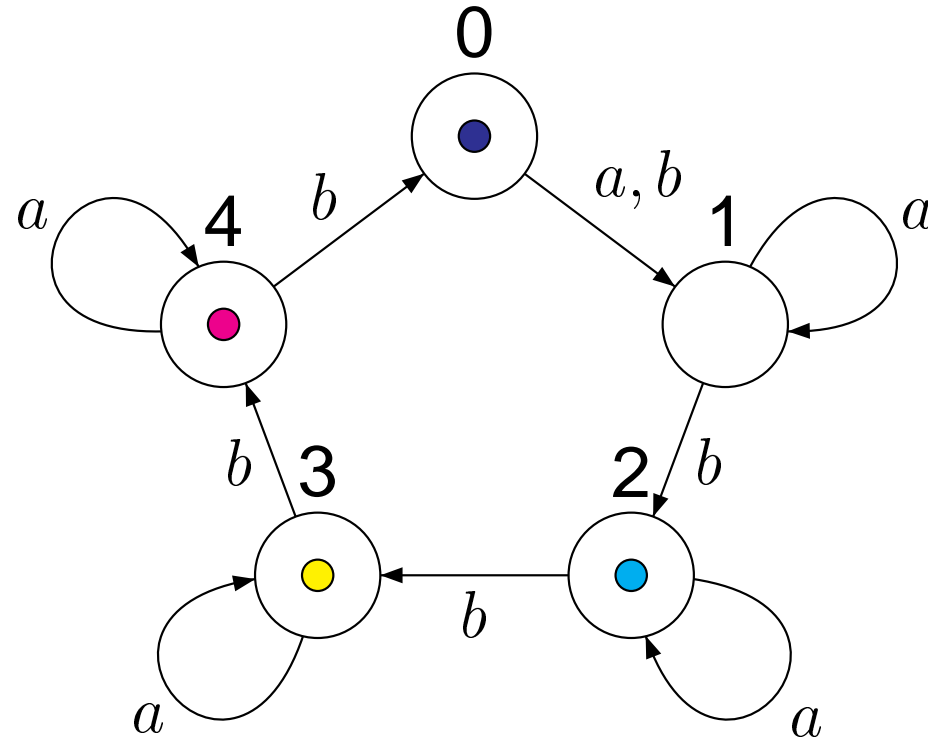
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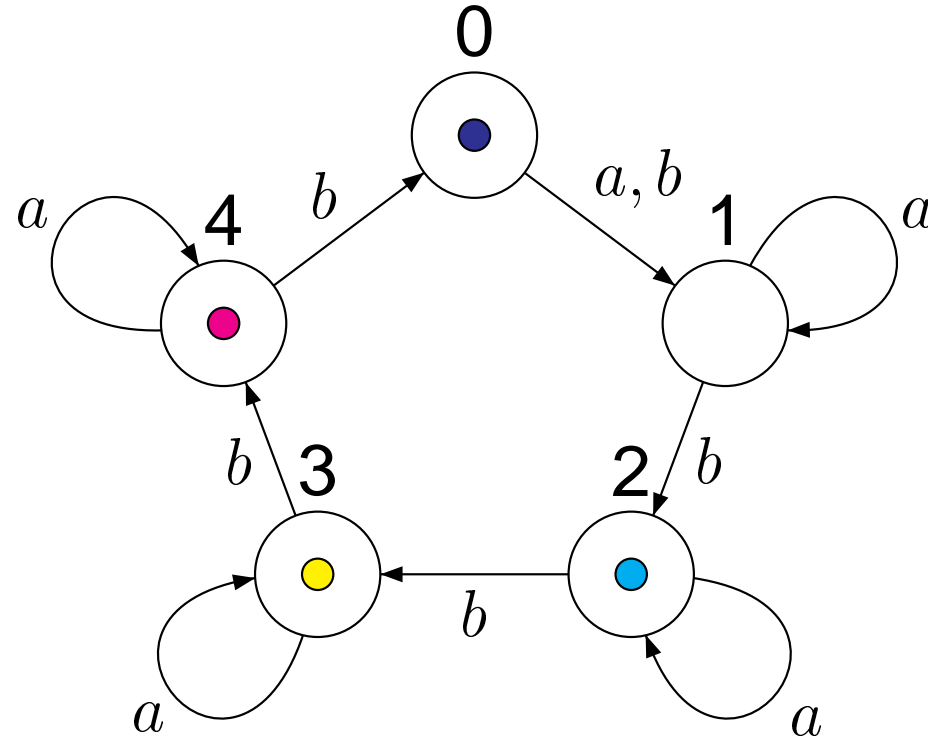


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We have to check that our weight function satisfies the conditions

- (i)  $\text{wg}(P_0) \geq n(n - 1)$  and  $\text{wg}(P_{|w|}) \leq n - 1$ ;
- (ii)  $1 \geq \text{wg}(P_{i-1}) - \text{wg}(P_i)$  for each  $i = 1, \dots, |w|$ .

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In the initial position all states are covered with coins. Consider the coin  $C$  that covers the state  $s_0(G) + 1 \pmod{n}$ , that is the state in one step clockwise after the state covered with the golden coin. Then  $d_0(C) = n - 1$  whence  $\text{wg}(C, P_0) = n \cdot (n - 1) + m_0(C) \geq n(n - 1)$ . Since the weight of a position is not less than the weight of any coin in this position, we have  $\text{wg}(P_0) \geq n(n - 1)$ .

In the final position only the golden coin  $G$  remains, whence the weight of  $P_{|w|}$  is the weight of  $G$ . Clearly,  $\text{wg}(G, P_i) = m_i(G) \leq n - 1$  for any position  $P_i$ .

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Let  $C$  be a coin of maximum weight in  $P_{i-1}$ . If the transition from  $P_{i-1}$  to  $P_i$  is caused by  $b$ , then  $d_i(C) = d_{i-1}(C)$  (because the relative location of the coins does not change) and  $m_i(C) = m_{i-1}(C) - 1$  if  $m_{i-1}(C) > 0$ , otherwise  $m_i(C) = n - 1$ .

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$$\begin{aligned} \text{wg}(P_i) &\geq \text{wg}(C, P_i) = n \cdot d_i(C) + m_i(C) \geq \\ n \cdot d_{i-1}(C) + m_{i-1}(C) - 1 &= \text{wg}(C, P_{i-1}) - 1 = \text{wg}(P_{i-1}) - 1. \end{aligned}$$

Suppose the transition from  $P_{i-1}$  to  $P_i$  is caused by  $a$ .

If  $s_{i-1}(C) \neq 0$ , then  $m_i(C) = m_{i-1}(C)$  and  $d_i(C) = d_{i-1}(C)$

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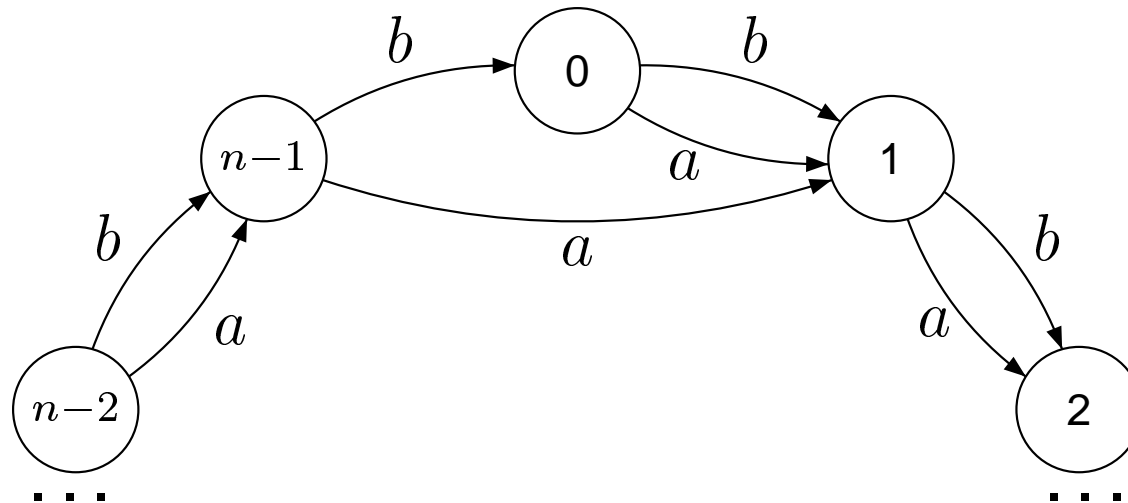
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it does not move and  $d_i(C') = d_{i-1}(C) - 1$ . Therefore

$$\begin{aligned} \text{wg}(P_i) &\geq \text{wg}(C', P_i) = n \cdot d_i(C') + n - 1 = n \cdot (d_{i-1}(C) - 1) + n - 1 \\ &= n \cdot d_{i-1}(C) - 1 = \text{wg}(C, P_{i-1}) - 1 = \text{wg}(P_{i-1}) - 1. \end{aligned}$$

# Ananichev's Example

The proof for Ananichev's example uses the same underlying idea but a more sophisticated weight function that is defined via pairs of coins.



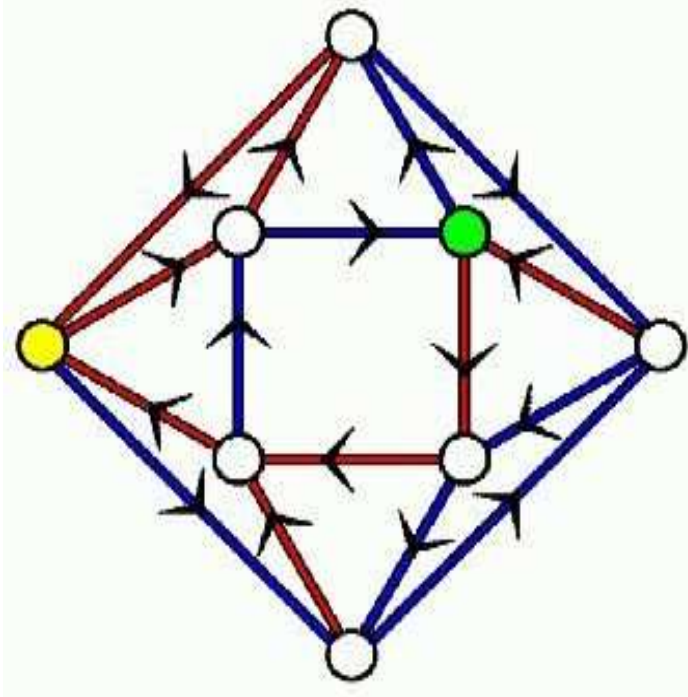
# *Road Coloring*

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We think of an automaton as of a scheme of a transport network in which arrows correspond to roads and labels are treated as colors of the roads.

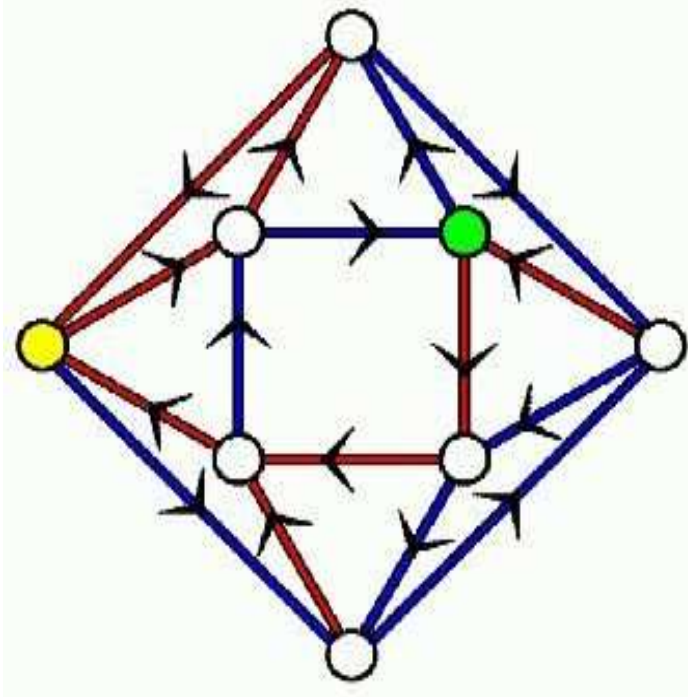
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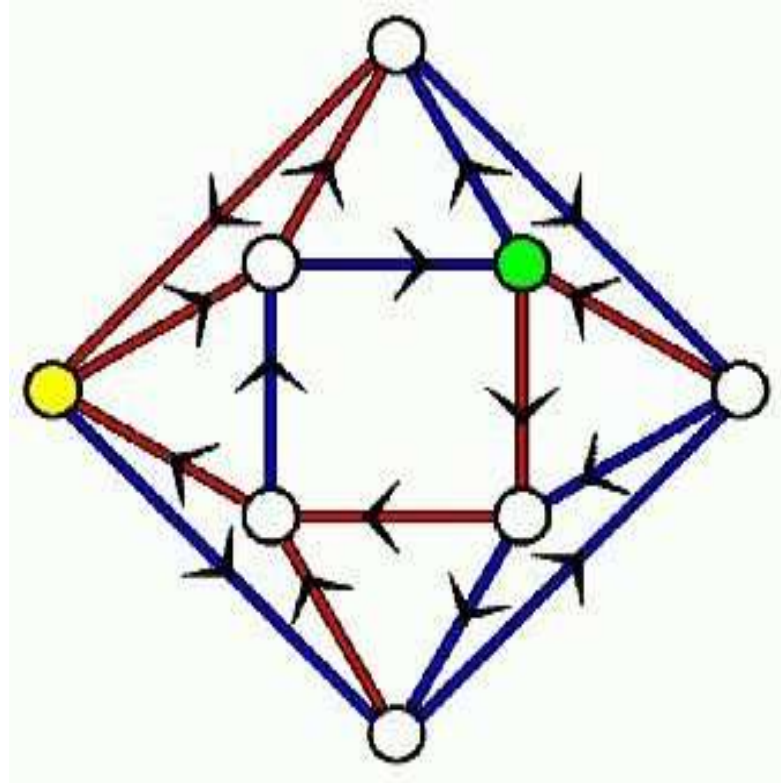
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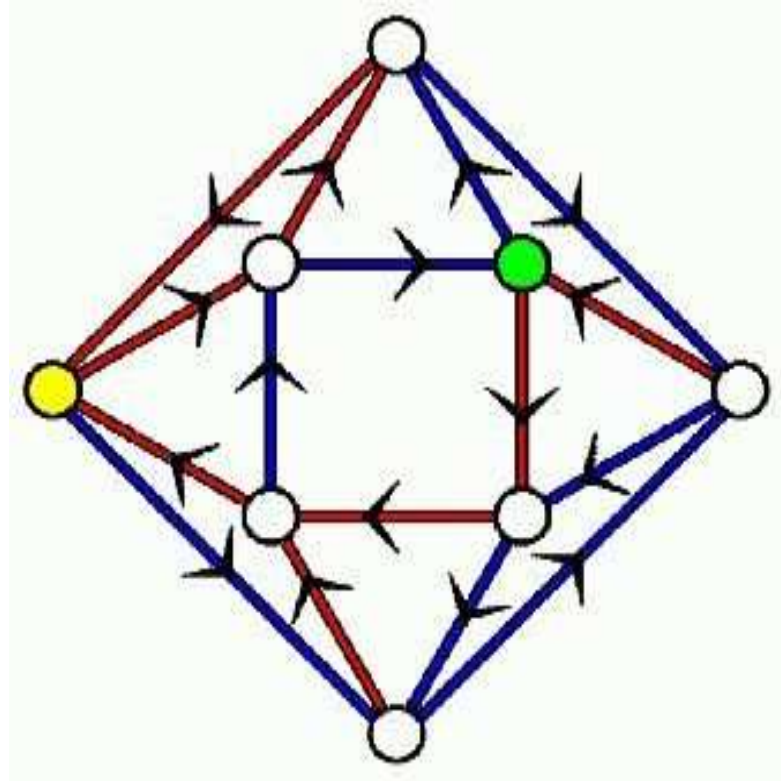


Then for each node  $N$  there is a sequence of colors that brings one to  $N$  from anywhere.

# Road Coloring

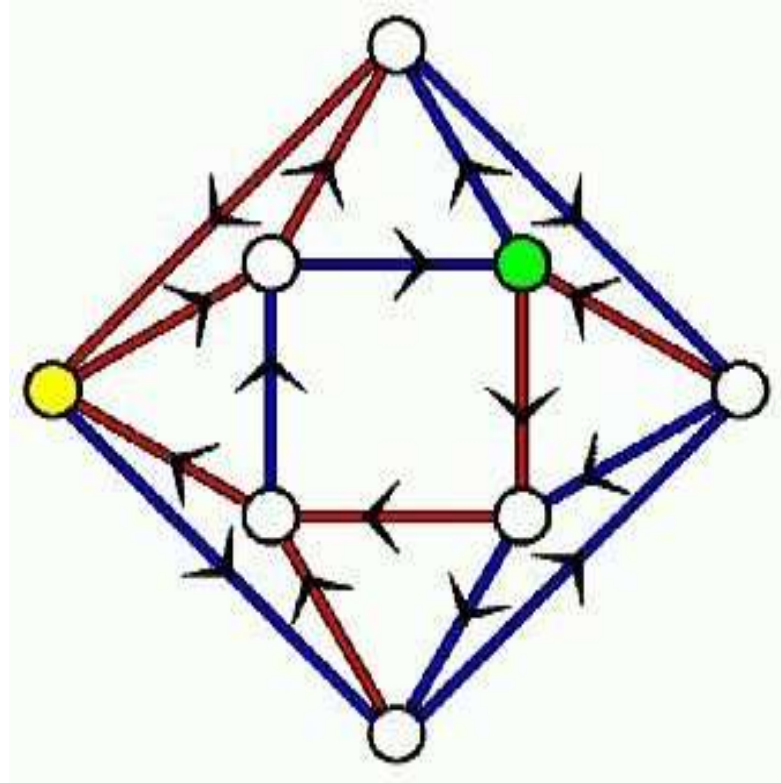


# Road Coloring



For the green node:  
blue-blue-red-blue-blue-red-blue-blue-red.

# Road Coloring



For the green node:  
blue-blue-red-blue-blue-red-blue-blue-red.

For the yellow node:  
blue-red-red-blue-red-red-blue-red-red.

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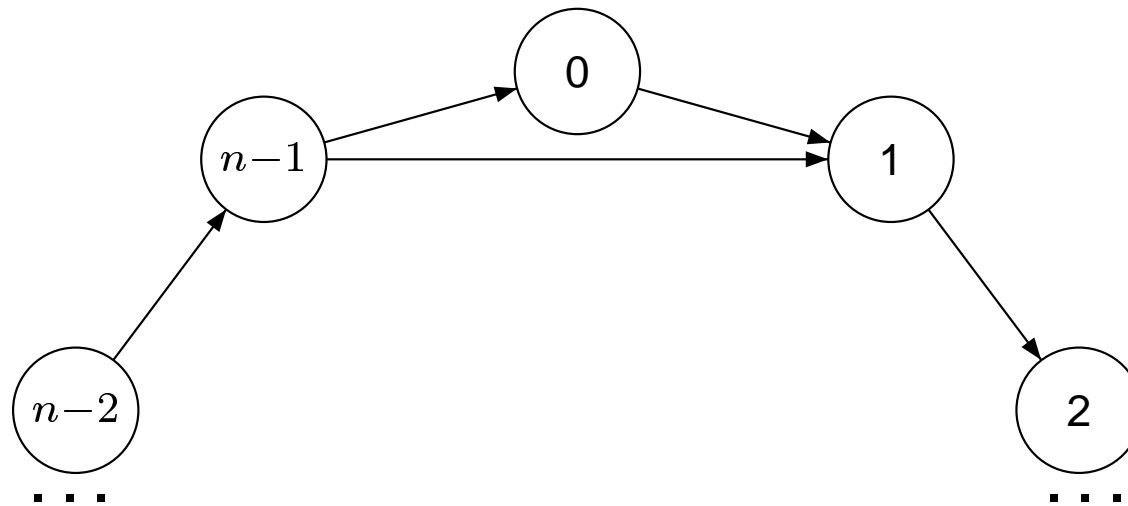
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Clearly, Trahtman's theorem holds true in this setting and may be stated as follows:

**every strongly connected primitive digraph admits a synchronizing coloring.**

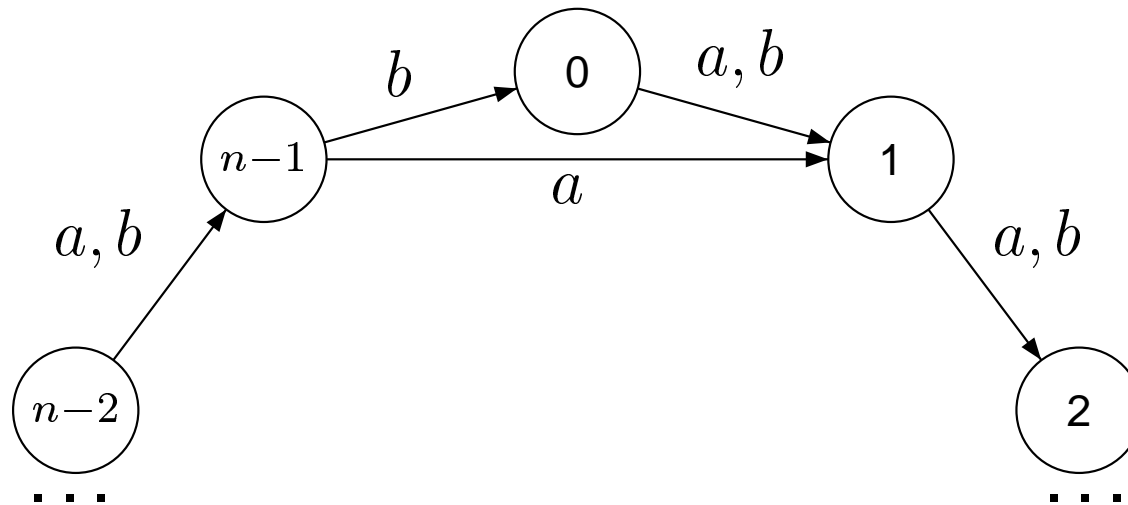
# Ananichev's Example

For an illustration, Ananichev's example can be obtained this way from the following digraph:



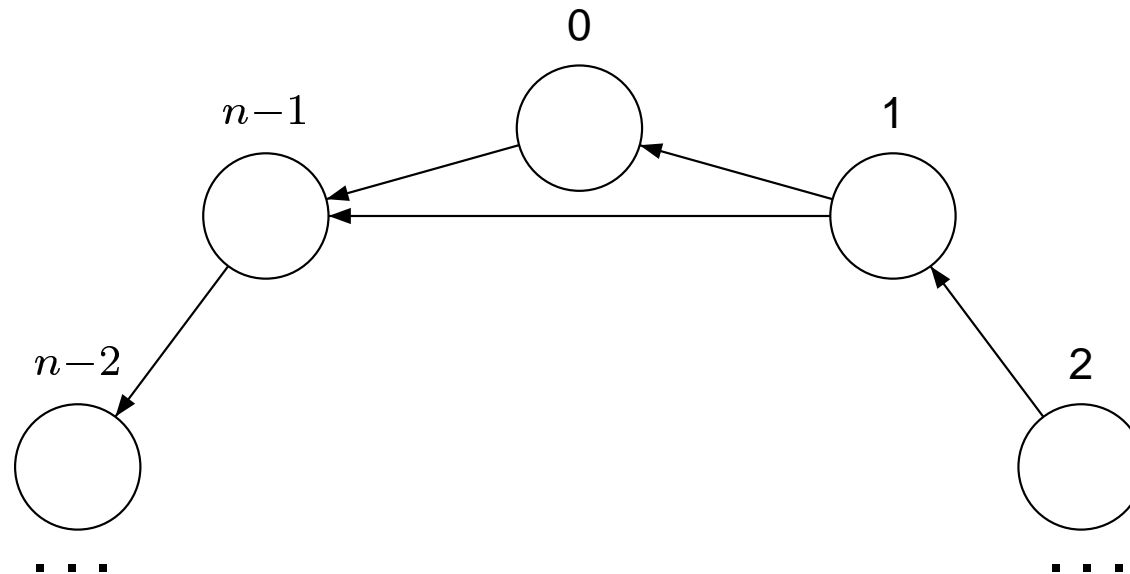
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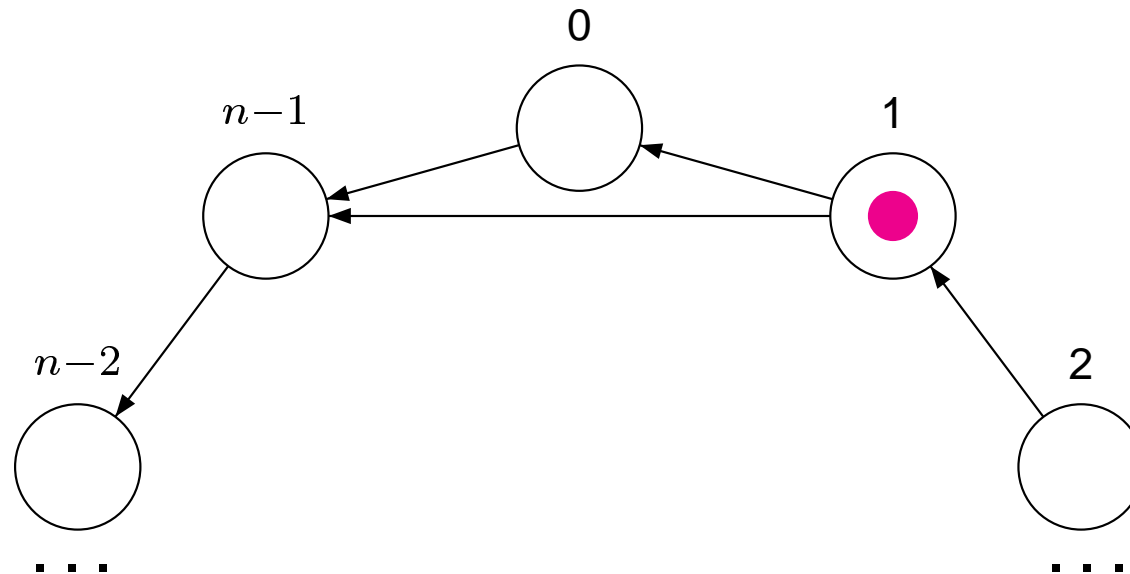
# Reverse Game

We can reverse the graph and play the reverse game in which only one vertex holds a coin in the beginning and coins multiply whenever sliding from a vertex with out-degree more than 1.



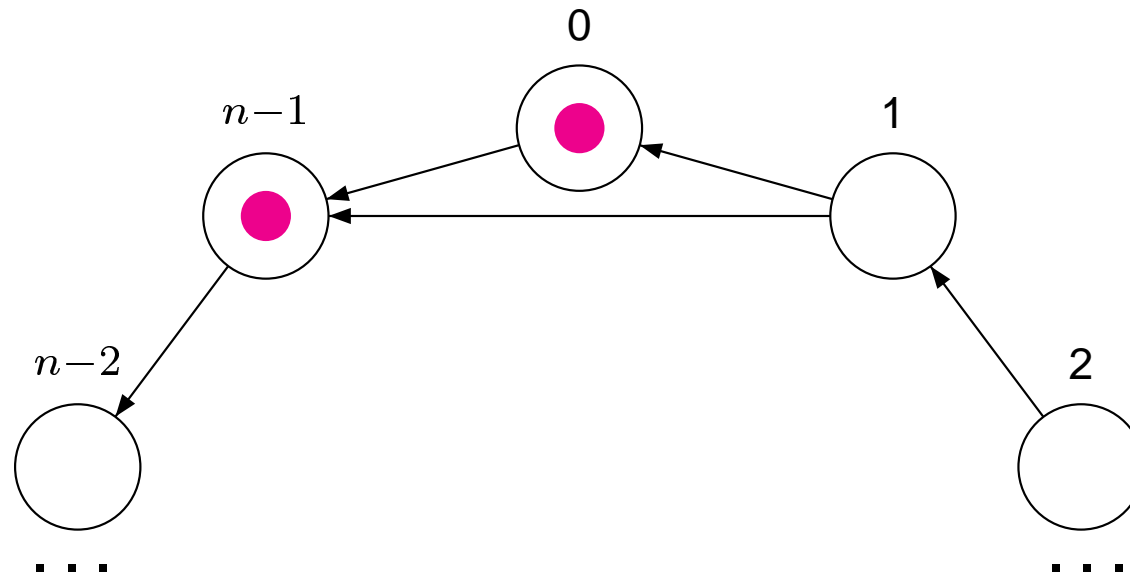
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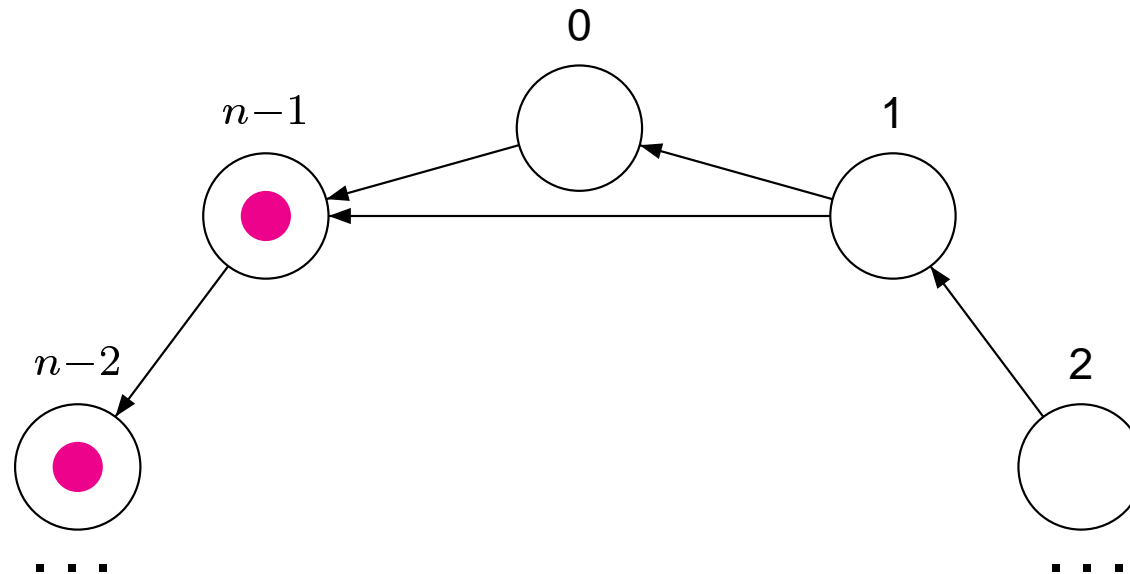
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## *Reverse Game*

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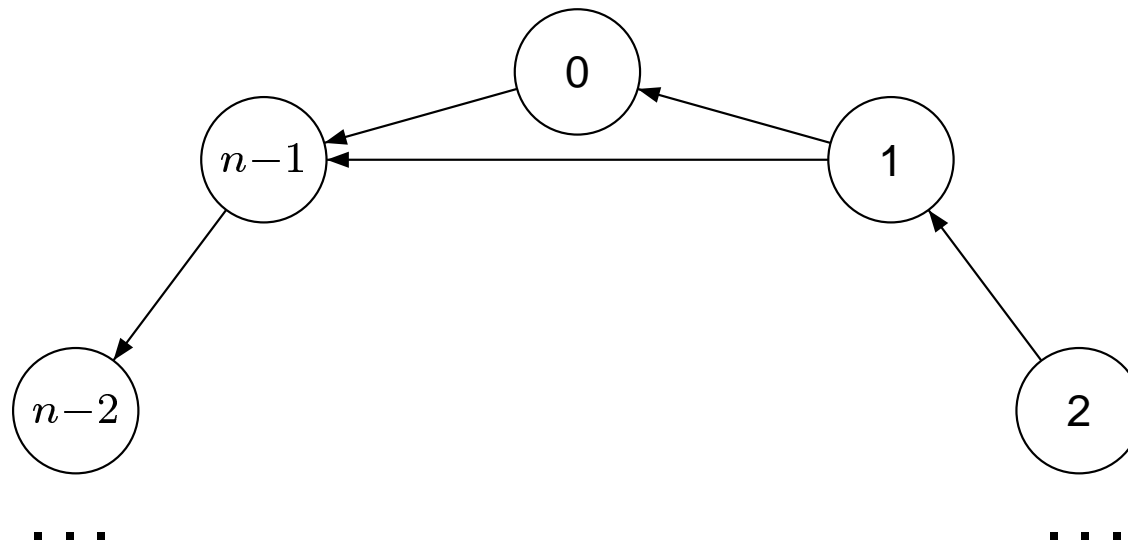
# *Wielandt's Theorem*

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We conjecture that  $n^2 - 3n + 3$  is the **exact value** in the hybrid Černý/Road Coloring problem. Though Wielandt's theorem doesn't immediately applies here, it gives a very strong supporting evidence for this conjecture as well as the original Černý conjecture.

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