The Finite Basis Problem for Hecke–Kiselman Monoids

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Operators from Convex Analysis – I

E – a real topological vector space

f – a function $E \to \mathbb{R} \cup \{+\infty, -\infty\}$

Three operators play a distinguished role in convex analysis

I. Convex hull or largest convex minorant $c(f)$

$=$ the supremum of all convex minorants of f.

Recall that a function $g\colon E \to \mathbb{R} \cup \{+\infty, -\infty\}$ is convex if its epigraph $\{(x, y) \mid y \geq f(x)\}$ is convex.

$$c(f)(x) =$$

$$= \inf\left\{ \sum_{i=1}^{N} \lambda_i f(x_i) \;\middle|\; N \geq 1,\ \lambda_i > 0,\ f(x_i) < +\infty,\ \sum_{i=1}^{N} \lambda_i x_i = x \right\}$$

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Operators from Convex Analysis – II and III

II. Largest lower semicontinuous minorant $\ell(f) =$ the supremum of all lower semicontinuous minorants of f with respect to the topology on E.

This corresponds to taking the closure of the epigraph of f with respect to the product of the topology on E and the usual topology on $\mathbb{R}$.

$$\ell(f)(x) = \liminf_{y \to x} f(y)$$

III. The operator m:

$$m(f)(x) = \begin{cases} f(x) & \text{if } f(x) > -\infty \text{ for all } x \in E; \\ -\infty & \text{otherwise} \end{cases}$$

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The operators c, ℓ, m generate a monoid $G(E)$ with composition as multiplication

Kiselman (A semigroup of operators in convexity theory, Trans. Amer. Math. Soc. 354 (2002), no. 5, 2035–2053) has shown that the order of $G(E)$ can be 1, 6, 15, 16, 17, or 18, depending on the dimension of E and its topology.

In particular, for every normed space E of infinite dimension, $G(E)$ consists of 18 elements and has the following presentation (as a monoid):

$$G(E) = \langle c, \ell, m \mid c^2 = c,\ \ell^2 = \ell,\ m^2 = m,\ c\ell c = \ell c\ell = \ell c,\ cmc = mcm = mc,\ \ell m\ell = m\ell m = m\ell\rangle$$

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General Kiselman Monoids

Ganyushkin and Mazorchuk have suggested to consider analogous presentations with arbitrarily many generators.

For each $n \geq 2$, they have defined the Kiselman monoid $\mathcal{K}_n$ as follows:

$$\mathcal{K}_n = \langle a_1, a_2, \dots, a_n \mid a_i^2 = a_i,\ i = 1, \dots, n; \\ a_i a_j a_i = a_j a_i a_j = a_j a_i,\ 1 \leq i < j \leq n \rangle.$$

Kiselman's monoid $G(E)$ is isomorphic to $\mathcal{K}_3$.

Algebraic properties of monoids $\mathcal{K}_n$ have been studied by Kudryavtseva and Mazorchuk (On Kiselman's semigroup, Yokohama Math. J. 55 (2009), no.1, 21–46). The monoid $\mathcal{K}_n$ is finite for every n but its exact order is not yet known.

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A further generalization has been suggested by Ganyushkin and Mazorchuk (On Kiselman quotients of 0-Hecke monoids, Int. Electron. J. Algebra 10 (2011), no. 2, 174–191).

Θ – an anti-reflexive binary relation on the set $\{1, 2, \ldots, n\}$, $n \geq 2$. The Hecke–Kiselman monoid $\mathcal{HK}_\Theta$ corresponding to Θ is the monoid generated by $a_1, a_2, \ldots, a_n$ subject to the relations

$$\begin{aligned}
a_i^2 &= a_i && \text{for each } i = 1, \ldots, n;\\
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$\mathcal{K}_n = \mathcal{HK}_\Theta$ for $\Theta := \Theta_K = \{(j,i) \mid 1 \leq i < j \leq n\}$

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Another important special case: Catalan monoid $\mathcal{C}_{n+1}$

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$\mathcal{C}_{n+1} = \mathcal{HK}_\Theta$ for $\Theta := \Theta_C = \{(i+1, i) \mid i = 1, 2, \dots, n-1\}$

Solomon: $\mathcal{C}_{n+1}$ can be identified with the monoid of all order-preserving and decreasing transformations of the chain $1 < 2 < \cdots < n < n+1$. A transformation α is decreasing if $i\alpha \leq i$ for each $i = 1, \dots, n+1$.

Higgins: $|\mathcal{C}_n|$ is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$

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Catalan Monoids as Universal Objects

Catalan monoids (in their incarnations as the monoids of order-preserving and decreasing transformations of chains) are known to be universal for the class of all finite $\mathscr{J}$-trivial monoids.

A monoid $\mathcal{M}$ is $\mathscr{J}$-trivial if every principal ideal of $\mathcal{M}$ has a unique generator, that is, $\mathcal{M}a\mathcal{M} = \mathcal{M}b\mathcal{M}$ implies $a = b$ for all $a, b \in \mathcal{M}$.

Finite $\mathscr{J}$-trivial monoids attract much attention because of their distinguished role in algebraic language theory (Simon) and representation theory (Denton *et al*).

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Catalan monoids (in their incarnations as the monoids of order-preserving and decreasing transformations of chains) are known to be universal for the class of all finite $\mathscr{J}$-trivial monoids. A monoid $\mathcal{M}$ is $\mathscr{J}$-trivial if every principal ideal of $\mathcal{M}$ has a unique generator, that is, $\mathcal{M}a\mathcal{M} = \mathcal{M}b\mathcal{M}$ implies $a = b$ for all $a, b \in \mathcal{M}$.

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A monoid $\mathcal{M}$ divides a monoid $\mathcal{N}$ if $\mathcal{M}$ is a homomorphic image of a submonoid in $\mathcal{N}$.

Identities

A semigroup identity is a pair of semigroup words (u, v) usually written as a formal equality $u \doteq v$.

A semigroup $\mathcal{S}$ satisfies an identity $u \doteq v$ (or: $u \doteq v$ holds in $\mathcal{S}$) if every evaluation of the letters involved in the words u and v at some elements of $\mathcal{S}$ produces equal values in $\mathcal{S}$.

Example: the identity $x^n \doteq x^{n+1}$ holds in the monoid $\mathcal{C}_{n+1}$. Indeed, evaluating x at any decreasing transformation α of the chain $1 < 2 < \cdots < n+1$, one sees that $i\alpha^n$ is a fixed point of α for each $i = 1, \ldots, n+1$, whence $i\alpha^n = i\alpha^{n+1}$ for all i, and thus, $\alpha^n = \alpha^{n+1}$.

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The Finite Basis Problem

The Finite Basis Problem (FBP) for a class **K** of algebras asks which algebras from **K** are finitely based and which are not.

Its algorithmic version is known as Tarski's problem.

For general algebras (and even for groupoids) the corresponding problem is shown by McKenzie to be undecidable.

For semigroups it remains open and attracts lots of attention and so does its restriction to the class of $\mathscr{J}$-trivial monoids.

See my survey "The finite basis problem for finite semigroups", Sci. Math. Jap., 53 (2001), no.1, 171–199; its (occasionally) updated version is available through my webpage:

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CAUL 34, November 21st, 2014

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X^* – the free monoid over an alphabet X.

A word $u = x_1 \cdots x_k$ with $x_1, \dots, x_k \in X$ is a scattered subword of $v \in X^*$ if $v = v_0 x_1 v_1 \cdots v_{k-1} x_k v_k$ for some $v_0, \dots, v_k \in X^*$.

Example: date is a scattered subword of derivative.

$w \sim_n w'$ iff w and w' share scattered subwords of length $\leq n$

Examples: $x^n \sim_n x^{n+1}$, $\quad xyzx^2tz \sim_3 xyxzx^2tz$.

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C_2, C_3, and C_4 are finitely based; C_n with $n \geq 5$ is nonfinitely based.

Recall: for an integer $n \geq 2$,

$$\Theta_K = \{(j, i) \mid 1 \leq i < j \leq n\},$$
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Theorem

Let $n \geq 2$. Then for every relation Θ on the set $\{1,2,\dots,n\}$ such that $\Theta_C \subseteq \Theta \subseteq \Theta_K$, the set of identities of the Hecke–Kiselman monoid $\mathcal{HK}_\Theta$ coincides with $\{w \doteq w' \mid w \sim_n w'\}$.

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Corollary

The Kiselman monoids $\mathcal{K}_2$ $(= \mathcal{C}_3)$ and $\mathcal{K}_3$ are finitely based; the monoid $\mathcal{K}_n$ with $n \geq 4$ is nonfinitely based.

Lemma (Ganyushkin and Mazorchuk)

Let Θ and Φ be anti-reflexive binary relations on the set $\{1, 2, \ldots, n\}$ and $\Phi \subseteq \Theta$. The map $a_i \mapsto a_i$ uniquely extends to a homomorphism from the Hecke–Kiselman monoid $\mathcal{HK}_\Theta$ onto the monoid $\mathcal{HK}_\Phi$.

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Recall: $\mathcal{HK}_\Theta$ is generated by $a_1, a_2, \ldots, a_n$ subject to the relations

$a_i^2 = a_i$	for each $i = 1, \ldots, n$;
$a_i a_j = a_j a_i$	if $(i, j), (j, i) \notin \Theta$;
$a_i a_j a_i = a_j a_i a_j$	if $(i, j), (j, i) \in \Theta$;
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Relations Between Hecke–Kiselman Monoids

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In particular, $\mathcal{C}_{n+1}$ is a homomorphic image of each monoid $\mathcal{HK}_\Theta$ with $\Theta_C \subseteq \Theta \subseteq \Theta_K$, and the latter is a homomorphic image of $\mathcal{K}_n$.

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Therefore, every identity $w \simeq w'$ that holds in the monoid $\mathcal{HK}_\Theta$ such that $\Theta_C \subseteq \Theta \subseteq \Theta_K$ must hold in $\mathcal{C}_{n+1}$ whence $w \sim_n w'$.

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To show that, conversely, every identity $w \doteq w'$ with $w \sim_n w'$ holds in each $\mathcal{HK}_\Theta$ such that $\Theta_C \subseteq \Theta \subseteq \Theta_K$, it suffices to verify that the identity holds in $\mathcal{K}_n$.

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d) If $u \sim_n v$, then there exists a word $w \in X^*$ that has each of the words u and v as a scattered subword and such that $u \sim_n w \sim_n v$.

Simplified Identity Basis

Corollary (Simplified basis)

For each $n \geq 1$, the collection of all identities of the form

$$u_k \cdots u_1 v_1 \cdots v_\ell \doteq u_k \cdots u_1 x v_1 \cdots v_\ell,$$

where $k, \ell \geq 0$, $k + \ell \geq n$, $x \in X$, and

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forms an identity basis for the set $\{w \doteq w' \mid w \sim_n w'\}$.

Indeed, if w'' is such that both w and w' are scattered subwords of w'' and $w \sim_n w'' \sim_n w'$, then the identity $w \doteq w'$ follows from $w \doteq w''$ and $w' \doteq w''$. Since w'' is obtained from w and w' by inserting letters, each of the latter identities follows from those of the form $uv \doteq uxv$ with $x \in X$ and $uxv \sim_n uv$.

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Indeed, if w'' is such that both w and w' are scattered subwords of w'' and $w \sim_n w'' \sim_n w'$, then the identity $w \doteq w'$ follows from $w \doteq w''$ and $w' \doteq w''$. Since w'' is obtained from w and w' by inserting letters, each of the latter identities follows from those of the form $uv \doteq uxv$ with $x \in X$ and $uxv \sim_n uv$.

Simplified Identity Basis

Corollary (Simplified basis)

For each $n \geq 1$, the collection of all identities of the form

$$u_k \cdots u_1 v_1 \cdots v_\ell \doteq u_k \cdots u_1 x v_1 \cdots v_\ell,$$

where $k, \ell \geq 0$, $k + \ell \geq n$, $x \in X$, and

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Combinatorics of Kiselman Monoids

$$\mathcal{K}_n = \langle a_1, a_2, \dots, a_n \mid a_i^2 = a_i,\ i = 1, \dots, n;$$
$$a_i a_j a_i = a_j a_i a_j = a_j a_i,\ 1 \leq i < j \leq n\rangle.$$

Elements of $\mathcal{K}_n$ are represented by words over $A_n = \{a_1, \dots, a_n\}$. Clearly, $c(w) = c(w')$ whenever $w, w' \in A_n^*$ represent the same element of $\mathcal{K}_n$. Thus, the content $c(s)$ of $s \in \mathcal{K}_n$ is well defined.

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Lemma (Reduction)

a) Let $s \in \mathcal{K}_n$ and $c(s) \subseteq \{a_i, a_{i+1}, \ldots, a_n\}$ for some i. Then $a_i s a_i = s' a_i$, where s' is obtained from s by removing all occurrences of a_i if there were some.

b) Let $t \in \mathcal{K}_n$ and $c(t) \subseteq \{a_1, \ldots, a_{i-1}, a_i\}$ for some i. Then $a_i t a_i = a_i t'$, where t' is obtained from t by removing all occurrences of a_i if there were some.

Lemma

$\mathcal{K}_n$ satisfies all identities of the form

$$u_k \cdots u_1 v_1 \cdots v_\ell \doteq u_k \cdots u_1 x v_1 \cdots v_\ell,$$

where $k, \ell \geq 0$, $k + \ell \geq n$, $x \in X$, and

$$c(u_k) \supseteq \cdots \supseteq c(u_1) \supseteq \{x\} \subseteq c(v_1) \subseteq \cdots \subseteq c(v_\ell).$$

Example: $\mathcal{K}_3$ satisfies $xy \cdot zx \cdot xtz \doteq xy \cdot x \cdot zx \cdot xtz$.
Consider an arbitrary evaluation $w \mapsto \overline{w}$. Then

$$c(\overline{xy}) \supseteq c(\overline{x}) \subseteq c(\overline{zx}) \subseteq c(\overline{xtz}).$$

If $a_1 \in c(\overline{xtz})$, we can remove a_1 from $\overline{x}$ and $\overline{zx}$ by the reduction lemma. But if $a_1 \notin c(\overline{xtz})$, then a_1 occurs in neither $\overline{x}$ nor $\overline{zx}$, etc.

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Open Questions - I

The monoids $G(E)$ for finite-dimensional E:

Label	Relation between $n = \dim E$ and $k = \dim \overline{\{0\}}$	Order of $G(E)$	The ideal I with $\mathcal{K}_3/I \cong G(E)$
A_1	$n = 0$	1	$\mathcal{K}_3$
A_{15}	$n = 1,\ k = 0$	15	$\{a_2a_3a_1, a_2a_3a_1a_2, a_3a_1a_2, a_3a_2a_1\}$
A_{16}	$n \geq 2,\ k = 0$	16	$\{a_2a_3a_1, a_2a_3a_1a_2, a_3a_2a_1\}$
B_6	$n = k > 0$	6	$\mathcal{K}_3a_2\mathcal{K}_3$
B_{16}	$n - 1 = k > 0$	16	$\{a_2a_3a_1a_2, a_3a_1a_2, a_3a_2a_1\}$
B_{17}	$n - 2 \geq k > 0$	17	$\{a_2a_3a_1a_2, a_3a_2a_1\}$

Open Questions - I, continued

Of course, A_1 is finitely based.

It can be easily shown that B_6 satisfies the same identities as $\mathcal{K}_2$ $(= \mathcal{C}_3)$ whence B_6 is also finitely based.

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Problem

Are the monoids A_{15}, A_{16}, B_{16}, *and* B_{17} *finitely based?*

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All these monoids satisfy the identity $x^2 \simeq x^3$ that fails in $\mathcal{K}_3$ hence our main result does not apply.
There exist finite monoids that satisfy all identities of $\mathcal{K}_3$ and are nonfinitely based:

$$\mathcal{L} = \langle e, f \mid e^2 = e,\ f^2 = f,\ efe = 0 \rangle.$$

Open Questions - II

Problem

Let Θ and Φ be anti-reflexive binary relations on the sets $V_n = \{1, 2, \ldots, n\}$ and respectively $V_m = \{1, 2, \ldots, m\}$. Under which necessary and sufficient conditions on the graphs (V_n, Θ) and (V_m, Φ) do the Hecke–Kiselman monoids $\mathcal{HK}_\Theta$ and $\mathcal{HK}_\Phi$ satisfy the same identities?

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Open Questions - III

An involuted semigroup is a semigroup equipped with a unary operation $s \mapsto s^*$ such that $(st)^* = t^*s^*$ and $(s^*)^* = s$ for all s, t.

Adding an involution may radically change the equational properties of a semigroup: a finitely based semigroup may become a nonfinitely based involuted semigroup and vice versa.

The the map $a_i \mapsto a_{n-i+1}$ uniquely extends to an involution in both $\mathcal{K}_n$ and $\mathcal{C}_{n+1}$.

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Do the Kiselman monoid $\mathcal{K}_n$ and the Catalan monoid $\mathcal{C}_{n+1}$ satisfy the same identities as involuted semigroups?

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Problem

a) For which n is $\mathcal{K}_n$ finitely based as an involuted semigroup?
b) For which n is $\mathcal{C}_n$ finitely based as an involuted semigroup?