

# The Finite Basis Problem for Hecke–Kiselman Monoids

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# Operators from Convex Analysis – I

$E$  – a real topological vector space

$f$  – a function  $E \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$

Three operators play a distinguished role in convex analysis

1. **Convex hull** or **largest convex minorant**  $c(f)$

= the supremum of all convex minorants of  $f$ .

Recall that a function  $g: E \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$  is **convex** if its **epigraph**  $\{(x, y) \mid y \geq f(x)\}$  is convex.

$$c(f)(x) = \inf \left\{ \sum_{i=1}^N \lambda_i f(x_i) \mid N \geq 1, \lambda_i > 0, f(x_i) < +\infty, \sum_{i=1}^N \lambda_i x_i = x \right\}$$

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II. **Largest lower semicontinuous minorant**  $\ell(f) =$   
the supremum of all lower semicontinuous minorants of  $f$   
with respect to the topology on  $E$ .

This corresponds to taking the closure of the epigraph of  $f$   
with respect to the product of the topology on  $E$  and the usual  
topology on  $\mathbb{R}$ .

$$\ell(f)(x) = \liminf_{y \rightarrow x} f(y)$$

III. The operator  $m$ :

$$m(f)(x) = \begin{cases} f(x) & \text{if } f(x) > -\infty \text{ for all } x \in E; \\ -\infty & \text{otherwise} \end{cases}$$

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The operators  $c, \ell, m$  generate a monoid  $G(E)$  with composition as multiplication

Kiselman (A semigroup of operators in convexity theory, Trans. Amer. Math. Soc. 354 (2002), no. 5, 2035–2053) has shown that the order of  $G(E)$  can be 1, 6, 15, 16, 17, or 18, depending on the dimension of  $E$  and its topology.

In particular, for every normed space  $E$  of infinite dimension,  $G(E)$  consists of 18 elements and has the following presentation (as a monoid):

$$G(E) = \langle c, \ell, m \mid c^2 = c, \ell^2 = \ell, m^2 = m, \\ clc = lcl = lc, cmc = mcm = mc, \ell m \ell = m \ell m = m \ell \rangle$$

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# General Kiselman Monoids

Ganyushkin and Mazorchuk have suggested to consider analogous presentations with arbitrarily many generators.

For each  $n \geq 2$ , they have defined the **Kiselman monoid**  $\mathcal{K}_n$  as follows:

$$\mathcal{K}_n = \langle a_1, a_2, \dots, a_n \mid a_i^2 = a_i, i = 1, \dots, n; \\ a_i a_j a_i = a_j a_i a_j = a_j a_i, 1 \leq i < j \leq n \rangle.$$

Kiselman's monoid  $G(E)$  is isomorphic to  $\mathcal{K}_3$ .

Algebraic properties of monoids  $\mathcal{K}_n$  have been studied by Kudryavtseva and Mazorchuk (On Kiselman's semigroup, Yokohama Math. J. 55 (2009), no.1, 21–46). The monoid  $\mathcal{K}_n$  is finite for every  $n$  but its exact order is not yet known.

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A further generalization has been suggested by Ganyushkin and Mazorchuk (On Kiselman quotients of 0-Hecke monoids, Int. Electron. J. Algebra 10 (2011), no. 2, 174–191).

$\Theta$  – an anti-reflexive binary relation on the set  $\{1, 2, \dots, n\}$ ,  $n \geq 2$ .  
The **Hecke–Kiselman monoid**  $\mathcal{HK}_\Theta$  corresponding to  $\Theta$  is the monoid generated by  $a_1, a_2, \dots, a_n$  subject to the relations

$$\begin{array}{ll} a_i^2 = a_i & \text{for each } i = 1, \dots, n; \\ a_i a_j = a_j a_i & \text{if } (i, j), (j, i) \notin \Theta; \\ a_i a_j a_i = a_j a_i a_j & \text{if } (i, j), (j, i) \in \Theta; \\ a_i a_j a_i = a_j a_i a_j = a_j a_i & \text{if } (i, j) \notin \Theta, (j, i) \in \Theta. \end{array}$$

$$\mathcal{K}_n = \mathcal{HK}_\Theta \text{ for } \Theta := \Theta_K = \{(j, i) \mid 1 \leq i < j \leq n\}$$

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Another important special case: **Catalan monoid**  $\mathcal{C}_{n+1}$

$\mathcal{C}_{n+1}$  is generated by  $a_1, a_2, \dots, a_n$  subject to the relations

$$\begin{aligned} a_i^2 &= a_i && \text{for each } i = 1, \dots, n; \\ a_i a_j &= a_j a_i && \text{if } |i - j| \geq 2, \quad i, j = 1, \dots, n; \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = a_{i+1} a_i && \text{for each } i = 1, \dots, n - 1. \end{aligned}$$

$\mathcal{C}_{n+1} = \mathcal{HK}_\Theta$  for  $\Theta := \Theta_C = \{(i + 1, i) \mid i = 1, 2, \dots, n - 1\}$

Solomon:  $\mathcal{C}_{n+1}$  can be identified with the monoid of all order-preserving and decreasing transformations of the chain  $1 < 2 < \dots < n < n + 1$ . A transformation  $\alpha$  is decreasing if  $i\alpha \leq i$  for each  $i = 1, \dots, n + 1$ .

Higgins:  $|\mathcal{C}_n|$  is the  $n$ -th Catalan number  $\frac{1}{n+1} \binom{2n}{n}$

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$$\begin{aligned} a_i^2 &= a_i && \text{for each } i = 1, \dots, n; \\ a_i a_j &= a_j a_i && \text{if } |i - j| \geq 2, \ i, j = 1, \dots, n; \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = a_{i+1} a_i && \text{for each } i = 1, \dots, n - 1. \end{aligned}$$

$$\mathcal{C}_{n+1} = \mathcal{HK}_\Theta \text{ for } \Theta := \Theta_C = \{(i + 1, i) \mid i = 1, 2, \dots, n - 1\}$$

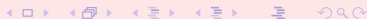
Solomon:  $\mathcal{C}_{n+1}$  can be identified with the monoid of all order-preserving and decreasing transformations of the chain  $1 < 2 < \dots < n < n + 1$ . A transformation  $\alpha$  is **decreasing** if  $i\alpha \leq i$  for each  $i = 1, \dots, n + 1$ .

Higgins:  $|\mathcal{C}_n|$  is the  $n$ -th Catalan number  $\frac{1}{n+1} \binom{2n}{n}$

# Catalan Monoids as Universal Objects

Catalan monoids (in their incarnations as the monoids of order-preserving and decreasing transformations of chains) are known to be universal for the class of all finite  $\mathcal{J}$ -trivial monoids. A monoid  $\mathcal{M}$  is  $\mathcal{J}$ -trivial if every principal ideal of  $\mathcal{M}$  has a unique generator, that is,  $Ma\mathcal{M} = Mb\mathcal{M}$  implies  $a = b$  for all  $a, b \in \mathcal{M}$ . Finite  $\mathcal{J}$ -trivial monoids attract much attention because of their distinguished role in algebraic language theory (Simon) and representation theory (Denton *et al*).

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Theorem (Pin, 1984, after Straubing, 1980)

*A finite monoid is  $\mathcal{J}$ -trivial iff it divides some Catalan monoid.*

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A monoid  $\mathcal{M}$  divides a monoid  $\mathcal{N}$  if  $\mathcal{M}$  is a homomorphic image of a submonoid in  $\mathcal{N}$ .

# Identities

A **semigroup identity** is a pair of semigroup words  $(u, v)$  usually written as a formal equality  $u \simeq v$ .

A semigroup  $\mathcal{S}$  **satisfies** an identity  $u \simeq v$  (or:  $u \simeq v$  **holds** in  $\mathcal{S}$ ) if every evaluation of the letters involved in the words  $u$  and  $v$  at some elements of  $\mathcal{S}$  produces equal values in  $\mathcal{S}$ .

*Example:* the identity  $x^n \simeq x^{n+1}$  holds in the monoid  $\mathcal{C}_{n+1}$ .  
Indeed, evaluating  $x$  at any decreasing transformation  $\alpha$  of the chain  $1 < 2 < \dots < n + 1$ , one sees that  $i\alpha^n$  is a fixed point of  $\alpha$  for each  $i = 1, \dots, n + 1$ , whence  $i\alpha^n = i\alpha^{n+1}$  for all  $i$ , and thus,  $\alpha^n = \alpha^{n+1}$ .

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# The Finite Basis Problem

The **Finite Basis Problem** (FBP) for a class  $\mathbf{K}$  of algebras asks which algebras from  $\mathbf{K}$  are finitely based and which are not.

Its algorithmic version is known as **Tarski's problem**.

For general algebras (and even for groupoids) the corresponding problem is shown by McKenzie to be undecidable.

For semigroups it remains open and attracts lots of attention and so does its restriction to the class of  $\mathcal{J}$ -trivial monoids.

See my survey "The finite basis problem for finite semigroups", Sci. Math. Jap., 53 (2001), no.1, 171–199; its (occasionally)

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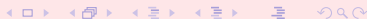
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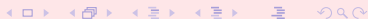
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# Identities of Catalan Monoids

$X^*$  – the free monoid over an alphabet  $X$ .

A word  $u = x_1 \cdots x_k$  with  $x_1, \dots, x_k \in X$  is a **scattered subword** of  $v \in X^*$  if  $v = v_0 x_1 v_1 \cdots v_{k-1} x_k v_k$  for some  $v_0, \dots, v_k \in X^*$ .

*Example:* **date** is a scattered subword of **derivative**.

$w \sim_n w'$  iff  $w$  and  $w'$  share scattered subwords of length  $\leq n$

*Examples:*  $x^n \sim_n x^{n+1}$ ,  $xyzx^2tz \sim_3 xyxzx^2tz$ .

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## Theorem ( $\sim$ , 2004)

*An identity  $w \doteq w'$  holds in the Catalan monoid  $\mathcal{C}_{n+1}$  iff  $w \sim_n w'$ .*

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Theorem ( $\sim$ , 2004)

*$\mathcal{C}_2$ ,  $\mathcal{C}_3$ , and  $\mathcal{C}_4$  are finitely based;  $\mathcal{C}_n$  with  $n \geq 5$  is nonfinitely based.*

# Identities of Hecke–Kiselman Monoids

Recall: for an integer  $n \geq 2$ ,

$$\Theta_K = \{(j, i) \mid 1 \leq i < j \leq n\},$$
$$\Theta_C = \{(i + 1, i) \mid i = 1, 2, \dots, n - 1\}.$$

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## Theorem

*Let  $n \geq 2$ . Then for every relation  $\Theta$  on the set  $\{1, 2, \dots, n\}$  such that  $\Theta_C \subseteq \Theta \subseteq \Theta_K$ , the set of identities of the Hecke–Kiselman monoid  $\mathcal{HK}_\Theta$  coincides with  $\{w \doteq w' \mid w \sim_n w'\}$ .*

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## Corollary

*The Kiselman monoids  $\mathcal{K}_2$  ( $= \mathcal{C}_3$ ) and  $\mathcal{K}_3$  are finitely based; the monoid  $\mathcal{K}_n$  with  $n \geq 4$  is nonfinitely based.*

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## Lemma (Ganyushkin and Mazorchuk)

*Let  $\Theta$  and  $\Phi$  be anti-reflexive binary relations on the set  $\{1, 2, \dots, n\}$  and  $\Phi \subseteq \Theta$ . The map  $a_i \mapsto a_i$  uniquely extends to a homomorphism from the Hecke–Kiselman monoid  $\mathcal{HK}_\Theta$  onto the monoid  $\mathcal{HK}_\Phi$ .*

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Recall:  $\mathcal{HK}_\Theta$  is generated by  $a_1, a_2, \dots, a_n$  subject to the relations

$$\begin{array}{ll} a_i^2 = a_i & \text{for each } i = 1, \dots, n; \\ a_i a_j = a_j a_i & \text{if } (i, j), (j, i) \notin \Theta; \\ a_i a_j a_i = a_j a_i a_j & \text{if } (i, j), (j, i) \in \Theta; \\ a_i a_j a_i = a_j a_i a_j = a_j a_i & \text{if } (i, j) \notin \Theta, (j, i) \in \Theta. \end{array}$$

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To show that, conversely, every identity  $w \simeq w'$  with  $w \sim_n w'$  holds in each  $\mathcal{HK}_\Theta$  such that  $\Theta_C \subseteq \Theta \subseteq \Theta_K$ , it suffices to verify that the identity holds in  $\mathcal{K}_n$ .

# Combinatorics of Scattered Subwords

$\sim_0$  – the universal relation on  $X^*$ ;  $c(w)$  – the **content** of  $w \in X^*$

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Let  $n \geq 1$ ,  $u, v \in X^*$ ,  $x \in X$ .

a)  $uxv \sim_n uv$  iff there exist  $k, \ell \geq 0$  with  $k + \ell \geq n$  and such that  $u \sim_k ux$  and  $xv \sim_\ell v$ .

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d) If  $u \sim_n v$ , then there exists a word  $w \in X^*$  that has each of the words  $u$  and  $v$  as a scattered subword and such that  $u \sim_n w \sim_n v$ .

# Simplified Identity Basis

## Corollary (Simplified basis)

For each  $n \geq 1$ , the collection of all identities of the form

$$u_k \cdots u_1 v_1 \cdots v_\ell \simeq u_k \cdots u_1 x v_1 \cdots v_\ell,$$

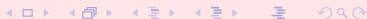
where  $k, \ell \geq 0$ ,  $k + \ell \geq n$ ,  $x \in X$ , and

$$c(u_k) \supseteq \cdots \supseteq c(u_1) \supseteq \{x\} \subseteq c(v_1) \subseteq \cdots \subseteq c(v_\ell)$$

forms an identity basis for the set  $\{w \simeq w' \mid w \sim_n w'\}$ .

Indeed, if  $w''$  is such that both  $w$  and  $w'$  are scattered subwords of  $w''$  and  $w \sim_n w'' \sim_n w'$ , then the identity  $w \simeq w'$  follows from  $w \simeq w''$  and  $w' \simeq w''$ . Since  $w''$  is obtained from  $w$  and  $w'$  by inserting letters, each of the latter identities follows from those of the form  $uv \simeq uxv$  with  $x \in X$  and  $uxv \sim_n uv$ .

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# Combinatorics of Kiselman Monoids

$$\mathcal{K}_n = \langle a_1, a_2, \dots, a_n \mid a_i^2 = a_i, i = 1, \dots, n; \\ a_i a_j a_i = a_j a_i a_j = a_j a_i, 1 \leq i < j \leq n \rangle.$$

Elements of  $\mathcal{K}_n$  are represented by words over  $A_n = \{a_1, \dots, a_n\}$ . Clearly,  $c(w) = c(w')$  whenever  $w, w' \in A_n^*$  represent the same element of  $\mathcal{K}_n$ . Thus, the content  $c(s)$  of  $s \in \mathcal{K}_n$  is well defined.

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## Lemma (Reduction)

- a) Let  $s \in \mathcal{K}_n$  and  $c(s) \subseteq \{a_i, a_{i+1}, \dots, a_n\}$  for some  $i$ . Then  $a_i s a_i = s' a_i$ , where  $s'$  is obtained from  $s$  by removing all occurrences of  $a_i$  if there were some.
- b) Let  $t \in \mathcal{K}_n$  and  $c(t) \subseteq \{a_1, \dots, a_{i-1}, a_i\}$  for some  $i$ . Then  $a_i t a_i = a_i t'$ , where  $t'$  is obtained from  $t$  by removing all occurrences of  $a_i$  if there were some.

## Lemma

$\mathcal{K}_n$  satisfies all identities of the form

$$u_k \cdots u_1 v_1 \cdots v_\ell \simeq u_k \cdots u_1 x v_1 \cdots v_\ell,$$

where  $k, \ell \geq 0$ ,  $k + \ell \geq n$ ,  $x \in X$ , and

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*Example:*  $\mathcal{K}_3$  satisfies  $xy \cdot zx \cdot xtz \simeq xy \cdot x \cdot zx \cdot xtz$ .

Consider an arbitrary evaluation  $w \mapsto \bar{w}$ . Then

$$c(\overline{xy}) \supseteq c(\bar{x}) \subseteq c(\overline{zx}) \subseteq c(\overline{xtz}).$$

If  $a_1 \in c(\overline{xtz})$ , we can remove  $a_1$  from  $\bar{x}$  and  $\overline{zx}$  by the reduction lemma. But if  $a_1 \notin c(\overline{xtz})$ , then  $a_1$  occurs in neither  $\bar{x}$  nor  $\overline{zx}$ , etc.

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The monoids  $G(E)$  for finite-dimensional  $E$ :

Label	Relation between $n = \dim E$ and $k = \dim \overline{\{0\}}$	Order of $G(E)$	The ideal $I$ with $\mathcal{K}_3/I \cong G(E)$
$A_1$	$n = 0$	1	$\mathcal{K}_3$
$A_{15}$	$n = 1, k = 0$	15	$\{a_2 a_3 a_1, a_2 a_3 a_1 a_2, a_3 a_1 a_2, a_3 a_2 a_1\}$
$A_{16}$	$n \geq 2, k = 0$	16	$\{a_2 a_3 a_1, a_2 a_3 a_1 a_2, a_3 a_2 a_1\}$
$B_6$	$n = k > 0$	6	$\mathcal{K}_3 a_2 \mathcal{K}_3$
$B_{16}$	$n - 1 = k > 0$	16	$\{a_2 a_3 a_1 a_2, a_3 a_1 a_2, a_3 a_2 a_1\}$
$B_{17}$	$n - 2 \geq k > 0$	17	$\{a_2 a_3 a_1 a_2, a_3 a_2 a_1\}$

Of course,  $A_1$  is finitely based.

It can be easily shown that  $B_6$  satisfies the same identities as  $\mathcal{K}_2 (= \mathcal{C}_3)$  whence  $B_6$  is also finitely based.

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There exist finite monoids that satisfy all identities of  $\mathcal{K}_3$  and are nonfinitely based:

$$\mathcal{L} = \langle e, f \mid e^2 = e, f^2 = f, efe = 0 \rangle.$$

## Problem

*Let  $\Theta$  and  $\Phi$  be anti-reflexive binary relations on the sets  $V_n = \{1, 2, \dots, n\}$  and respectively  $V_m = \{1, 2, \dots, m\}$ . Under which necessary and sufficient conditions on the graphs  $(V_n, \Theta)$  and  $(V_m, \Phi)$  do the Hecke–Kiselman monoids  $\mathcal{HK}_\Theta$  and  $\mathcal{HK}_\Phi$  satisfy the same identities?*

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An **involuted semigroup** is a semigroup equipped with a unary operation  $s \mapsto s^*$  such that  $(st)^* = t^*s^*$  and  $(s^*)^* = s$  for all  $s, t$ . Adding an involution may radically change the equational properties of a semigroup: a finitely based semigroup may become a nonfinitely based involuted semigroup and vice versa. The the map  $a_i \mapsto a_{n-i+1}$  uniquely extends to an involution in both  $\mathcal{K}_n$  and  $\mathcal{C}_{n+1}$ .

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## Problem

- a) For which  $n$  is  $\mathcal{K}_n$  finitely based as an involuted semigroup?*
- b) For which  $n$  is  $\mathcal{C}_n$  finitely based as an involuted semigroup?*