Relatively inherently non-finitely based varieties and quasi-varieties

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AAA82, June 25, 2011

Definitions and Terminology

A semigroup identity is a pair of semigroup words (u, v) usually written as a formal equality $u \bumpeq v$.

A semigroup S satisfies an identity $u \bumpeq v$ (or: $u \bumpeq v$ holds in S) if every evaluation of letters involved in the words u and v at some elements of S produces equal values in S.

Example: the identities $xy \bumpeq yx$ and $x \bumpeq x^2$ hold in the semigroup $\langle\{0, 1\}; \cdot\rangle$ while the identity $x \bumpeq y$ does not.

A semigroup S is finitely based if all identities holding in S can be deduced from some finite set of such identities (called an identity basis for S).

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If an identity $u \doteq v$ holds in $\langle\{0,1\};\cdot\rangle$, then u and v have the same letters. Indeed, if u has a letter, x, say, that does not occur in v, we evaluate x at 0 and all other letters at 1. Then the value of u becomes 0 while the value of v is 1.

On the other hand, the laws $xy \doteq yx$ and $x \doteq x^2$ allow one to reduce any word to the product of its letters, each taken once, in some fixed order.

$$xyxyzytzx \xrightarrow{xy\doteq yx} x^3y^3z^2t \xrightarrow{x\doteq x^2} xyzt$$

Hence, whenever two words involve the same letters, they can be reduced to the same product, and thus, to each other.

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Perkins's Example

If a semigroup is not finitely based, it is said to be nonfinitely based.

A finite semigroup can be nonfinitely based (Perkins, 1966). Perkins's example is the 6-element monoid B_2^1 (the Brandt monoid) formed by the following 2×2-matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The example is extremely transparent and natural. It also turns out to be minimal with respect to size of semigroup (Trahtman, 1981, 1991; Lee, 2011.) There are exactly three other 6-element examples (the examples are due to Trahtman, $\sim$, Zhang while the verification that the remaining 15969 semigroups of order 6 are finitely based is due to Lee & Li).

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Tarski's Problem

Semigroups are the only "classical" algebras for which finite non-finitely based objects exist: finite groups, associative or Lie rings, lattices are finitely based.

Problem: Which finite semigroups are finitely based and which are not?

This is the Finite Basis Problem (FBP) for finite semigroups.
Its algorithmic version is known as Tarski's problem for semigroups.

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Groupoids	Semigroups	Groups
Undecidable (McKenzie, 1996)		

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Why does Tarski's Problem cause so much interest?

Because it **is** interesting! As David Hilbert said, Mathematics is "eine einzigartige Symphonie des Unendlichen", i.e. a unique symphony of the infinite. The fact that the infinite may be hidden inside apparently finite and innocently looking objects such as the the 6-element Brandt monoid is surprising and challenging. It is something "wir müssen wissen" (we must know), as the same Hilbert said.

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FBP vs VMP

The Variety Membership Problem (VMP) for a finite semigroup S is a combinatorial decision problem whose input is a finite semigroup T and whose question is whether T belongs to the variety $\operatorname{var} S$.

VMP and related problems for pseudovarieties play a central role in the modern theory of finite semigroups.

VMP is always decidable: by the HSP-theorem $T \in \operatorname{var} S$ iff T is a homomorphic image of the free $|T|$-generated semigroup of $\operatorname{var} S$ and the free semigroup has at most $|S|^{(|S|^{|T|})}$ elements.

The complexity of VMP is not known (for general algebras it can be extremely high as shown by Kozik).

If S has a finite identity basis Σ, say, then in order to check whether or not T belongs to $\operatorname{var} S$ it suffices to verify whether or not all identities in Σ hold in T, and this is a polynomial (in $|T|$) procedure.

This gives an additional strong motivation for studying the FBP for finite semigroups.

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Quasi-identities

A semigroup quasi-identity is an expression of the form

$$u_1 \doteq v_1 \ \& \ \cdots \ \& \ u_n \doteq v_n \to u \doteq v, \qquad (*)$$

where $u_1, v_1, \dots, u_n, v_n, u, v$ are semigroup words.

A semigroup S satisfies a quasi-identity (*) (or: (*) holds in S) if every evaluation of letters involved in the words $u_1, v_1, \dots, u_n, v_n$, u, v at some elements of S produces equal values of u and v whenever values of u_i and v_i are equal for each $i = 1, \dots, n$.

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$$x^2 \doteq x \ \& \ y^2 \doteq y \to (xy)^2 \doteq xy$$

holds in the Brandt monoid B_2^1 while the quasi-identity

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Further Examples

Standard examples of semigroup quasi-identities are the cancellation laws

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There are many other important properties described by quasi-identities. For example, the quasi-identity

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Nonfinitely q-Based Semigroups

A semigroup S is finitely q-based if all quasi-identities holding in S can be deduced from some finite set of such quasi-identities (called an quasi-identity basis for S).

If a semigroup is not finitely q-based, it is said to be nonfinitely q-based.

A finite semigroup (even a finite group) can be nonfinitely q-based (Ol'shanskiĭ, 1974).

The 8-element group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ may serve as an example (in fact, this is a minimum size group example). A minimum size semigroup example is due to Sapir (1980): the 3-element cyclic semigroup $C_{3,1} = \langle a \mid a^3 = a^4 \rangle = \{a, a^2, 0\}$.

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AAA82, June 25, 2011

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A semigroup [quasi-]variety **V** is locally finite if every finitely generated semigroup in **V** is finite. Every [quasi-]variety generated by a finite semigroup S is locally finite: the largest n-generated semigroup of var S has at most $|S|^{(|S|^n)}$ elements.

A finite semigroup S is said to be inherently nonfinitely based if S is not contained in any locally finite finitely based variety. Hence S is nonfinitely based (and even inherently nonfinitely based) if var S contains an inherently nonfinitely based semigroup.

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Analogously, one can define inherently nonfinitely *q*-based semigroups as finite semigroups that are not contained in any locally finite finitely *q*-based quasi-variety.

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In particular, Sapir has shown that the 6-element Brandt monoid B_2^1

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Though very powerful, Sapir's results do not explain all "mass" examples of nonfinitely based finite semigroups.

For instance, it turns out (∼, 2004) that for all $n > 5$ the monoids of all extensive order preserving transformations of the chain $1 < 2 < \cdots < n$ are nonfinitely based though none of them are inherently nonfinitely based. (A transformation α is extensive if $i\alpha \geq i$ for every i.)

There are many results of such flavor.

Problem: Is there a common reason that makes all "sufficiently large" transformation monoids be nonfinitely based?

I strongly believe that such a common mechanism exists and that understanding it may constitute a major progress towards the solution of Tarski's problem.

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INFQB Semigroups

For semigroups, the INFQB property turns out to be too restrictive.

Margolis and Sapir in 1995 showed that there are no INFQB finite semigroups. In other words, every finite semigroup is contained in a locally finite finitely q-based quasi-variety.

(In contrast, Lawrence and Willard in 1998 constructed some finite INFQB unary semigroups, that is, semigroups equipped with an additional unary operation).

Thus, we look for a suitable relaxation of the concepts of being inherently nonfinitely based or inherently nonfinitely q-based.

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We suggest the following general framework in which the notions of being inherently nonfinitely q-based fit rather naturally.

Let $\mathscr{P}$ be a property of quasi-varieties of semigroups, such as "local finiteness", or "being generated by a finite set of finite semigroups".

The usual notion of INFQB coincides with INFQB with respect to local finiteness. When $\mathscr{P}$ is the property of being generated by a finite set of finite semigroups, we obtain another notion that has been studied in the literature, namely, the notion of strongly non-finitely q-based quasi-variety.

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- **K** generates a quasi-variety satisfying $\mathscr{P}$ and
- whenever a class of semigroups **J** generates a quasi-variety satisfying $\mathscr{P}$ and containing **K**, then **J** has no finite basis of quasi-identities.

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Let $\mathscr{P}$ be a property of quasi-varieties of semigroups, such as "local finiteness", or "being generated by a finite set of finite semigroups". A class **K** of semigroups is inherently nonfinitely q-based relative to $\mathscr{P}$ (abbreviated INFQB relative to $\mathscr{P}$) if

- **K** generates a quasi-variety satisfying $\mathscr{P}$ and
- whenever a class of semigroups **J** generates a quasi-variety satisfying $\mathscr{P}$ and containing **K**, then **J** has no finite basis of quasi-identities.

The usual notion of INFQB coincides with INFQB with respect to local finiteness. When $\mathscr{P}$ is the property of being generated by a finite set of finite semigroups, we obtain another notion that has been studied in the literature, namely, the notion of strongly non-finitely q-based quasi-variety.

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Examples of Relatively INFQB Semigroups

Every proper 3-nilpotent semigroup and every finite semigroup containing an element a such that $a^2 \neq a^3$ is INFQB relative to each of the following properties:

This shows that virtually any semigroup in a "nice" way (as injective partial maps or as order preserving maps) admit no "nice" quasi-identity basis, and also that two natural finiteness conditions—finite axiomatisability and finite generation—are virtually incompatible.

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A Corollary

For some property $\mathscr{P}$ of algebras, and a variety $\mathbf{V}$ let $\mathscr{P}_{\mathbf{V}}(n)$ denote the fraction of all $\mathbf{V}$-algebras on the set $\{0, 1, \ldots, n-1\}$ with property $\mathscr{P}$, if it exists, and let $\mathscr{P}_{\mathbf{V}}(n) = 1$ otherwise.

Let $\mathscr{F}\mathscr{B}$ denote finite basis property and let $\mathscr{F}\mathscr{Q}\mathscr{B}$ denote the finite q-basis property. Murskiĭ in 1975 showed that

$$\lim_{n\to\infty} \mathscr{F}\mathscr{B}_{\mathbf{V}_{\mathcal{F}}}(n) = 1,$$

when $\mathbf{V}_{\mathcal{F}}$ is the variety of all algebras of some finite signature $\mathcal{F}$. In this sense, almost all finite algebras are finitely based.

The corresponding results

$$\lim_{n\to\infty} \mathscr{F}\mathscr{B}_{\mathrm{Sgp}}(n) = \lim_{n\to\infty} \mathscr{F}\mathscr{B}_{\mathrm{Mon}}(n) = 1$$

for the variety Sgp of all semigroups and the variety Mon of all monoids (in the type $\langle 2, 0\rangle$) also hold true; so semigroups and monoids behave quite “typically” with respect to the axiomatisability properties of their identities.

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A Corollary Continued

In fact, Murskiĭ's results also imply that

$$\lim_{n\to\infty} \mathscr{F}\mathscr{Q}\mathscr{B}_{\mathbf{V}_{\mathcal{F}}}(n) = 1,$$

provided $\mathcal{F}$ contains an operation of arity more than 1. Thus, almost all finite algebras are finitely q-based.

In contrast, our result shows that semigroups and monoids are in fact atypical with respect to satisfaction of the finite q-basis property:

$$\lim_{n\to\infty} \mathscr{F}\mathscr{Q}\mathscr{B}_{\mathsf{Sgp}}(n) = \lim_{n\to\infty} \mathscr{F}\mathscr{Q}\mathscr{B}_{\mathsf{Mon}}(n) = 0.$$

This immediately follows from the known fact that almost all finite semigroups [monoids] are proper 3-nilpotent semigroups [proper 3-nilpotent semigroups with 1 adjoined].

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Examples of Relatively INFB Algebras

So far we have no striking example of a finite semigroup which is INFB relative to some interesting property.

However, I strongly recommend the paper "The equational complexity of Lyndon's algebra" by Marcel Jackson and George McNulty (Algebra Universalis Volume 65, Number 3 (2011), 243–262) where they show that Lyndon's 7-element groupoid of 1954—the earliest example of a nonfinitely based finite algebra—is INFB relative to the property to be contained in the variety generated by all automatic algebras.

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Conclusion

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AAA82, June 25, 2011

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Let us all meet at AAA-100!